

# Modelling Lateral Spread in Wire Flat Rolling

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## Abstract

A mathematical model for wire rolling is developed and compared against experiments, focusing on predicting the lateral spread. This provides, for the first time, an analytic model of lateral spread without any fitting parameters. The model is derived directly from the governing equations; in order to obtain a simple model, a rigid perfectly plastic material is assumed and simplified with the assumption of small thickness and width compared to the length of the roll gap (effectively a thin-wire large-roller assumption). The model is compared against experiments performed on stainless steel wire using 100 mm diameter rolls, and is shown to provide accurate predictions of lateral spread across a wide range of wire diameters (2.96 mm–7.96 mm) and reduction ratios (20%–60%), all without the need for fitting parameters. The model requires only seconds to compute and serves as a robust tool for validating FE results, guiding process design, and laying the foundation for future improved models. MATLAB code to evaluate the model is provided in the supplementary material.

**Keywords:** mathematical modelling, plastic deformation, flat rolling of wire, lateral spread, quick-to-compute

## 1. Introduction

Applications including sawblades, springs, piston rings, and transformers depend critically on flattened wires [1]. These wires are usually made by a flat rolling technique where a wire with a circular cross-section is cold rolled between cylindrical rolls — sometimes through several passes — to achieve a particular width and thickness (figure 1). Since the wire can both elongate and widen, achieving a final product that closely matches the desired specifications requires an understanding of how the wire deforms, making it essential to predict the lateral spread accurately.

In cold rolling of a sheet, deformation in the lateral direction often remains within the elastic range due to the product's geometry. However, this is not the case when an initial width-to-thickness ratio ( $W_0/2h_0$ ) is less than 6, such as in wide strip or plate, or as low as 1 in round wire [3]. In such cases, the plastic flow in the roll gap at the start of rolling schedule is inherently three-dimensional, complicating the analysis. For round wire, this complexity is further increased by the transformation of a round cross-section into a rectangular shape with bulged edges during the first pass.

Although research exists on the modelling of lateral spread in thick plate rolling [e.g. 3–7], wire flat rolling has received less attention. Kazeminezhad and Karimi

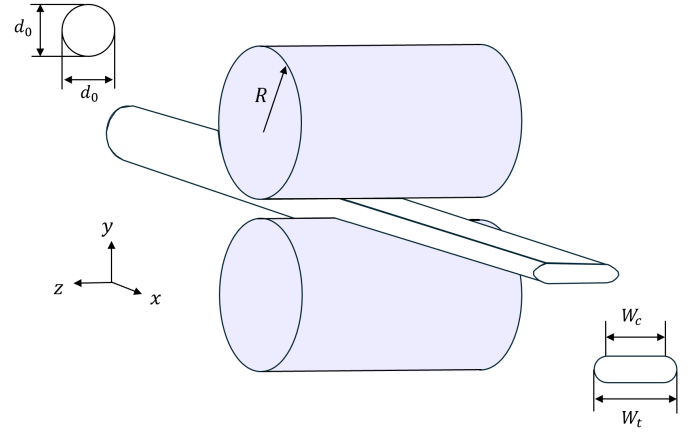


Figure 1: A diagram of wire flat rolling process; the wire initially has a circular cross-section with diameter  $d_0$  and is flattened to a barrel shape cross section with the lateral spread  $W_t$  and the contact width  $W_c$ . Adapted from Figure 1 of Carlsson [2].

Taheri published a series of papers studying different parameters in wire rolling [8–10]. Initially, Kazeminezhad and Karimi Taheri [8] developed a relationship for the width of the contact area between the rolls and the wire. Their equation was based on experimental and FE observations of the formation of X-shaped shear bands in the wire cross-section during the rolling process [11, 12], along with the assumption that as the height reduction increases, the shear bands rotate while maintaining a constant length. This equation was then applied alongside the slab method [9], leading to a pressure hill distribution. The study employs the Tresca yield criterion, assuming

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the longitudinal stress  $\sigma_{xx}$  to be the minimum stress, the vertical stress  $\sigma_{yy}$  to be the maximum stress, and the compressive stress in the lateral direction,  $\sigma_{zz}$ , to lie between them. Although  $\sigma_{zz}$  is implicitly included in the yield function, and therefore in calculating the roll pressure, the value of  $\sigma_{zz}$  is unknown and must be approximated to find the lateral spread. Kazeminezhad and Karimi Taheri [8] initially assumed  $\sigma_{zz}$  to take a value between plane strain and plane stress, written as  $m\sigma_{yy}$ , where  $m$  is a fitting parameter taking values between 0 and 0.5. Using flow rule equations and setting  $\sigma_{xx}$  to be zero at the end of the roll gap, they then wrote

$$\frac{d\varepsilon_{zz}}{-d\varepsilon_{yy}} = \frac{\ln \frac{W_t}{d_0}}{\ln \frac{d_0}{2h_1}} = \frac{1 - 2m}{2 - m}, \quad (1)$$

where  $2h_1$  is the final height,  $d_0$  is the initial diameter of the wire, and  $W_t$  is the lateral spread of the wire. The first use of this formulation has been attributed to Hill [13] for plate rolling and has also been derived by several other authors [e.g. 3, 14], who experimentally observed a linear relationship between  $\ln(W_t/W_0)$  and  $\ln(h_0/h_1)$ . The factor  $m$  and another factor of linearity must then be found empirically from experiments, with a proviso that each resulting equation only gives reasonable lateral spread prediction within the ranges of conditions for which they were empirically determined [3]. For example, for low and high carbon steel Kazeminezhad and Karimi Taheri [8] wrote equation (1) as

$$\frac{W_t}{d_0} = 1.02 \left( \frac{d_0}{2h_1} \right)^{0.45}. \quad (2)$$

Among a wide range of empirical formulations suggested for predicting lateral spread [3], the one proposed by Kobayashi [15] is often referred to as being reliable for round wire [1, 10, 16]. Utsunomiya et al. [1] and Vallellano et al. [16] attribute the following equation to Kobayashi [15], and we will refer to it here by this name:

$$\frac{W_t}{d_0} = 0.7854 \frac{d_0}{2h_1} \left( 1 - 15.8 \left( 1 - \frac{2h_1}{d_0} \right)^{2.25} \left( \frac{2R}{d_0} \right)^{-0.82} \right) + 0.1426 \left( \frac{2h_1}{d_0} \right), \quad (3)$$

where  $R$  is the roll radius. However, similar to other empirical equations, this equation will be shown to lose accuracy away from the parameters it was initially developed to model.

Similar to other metal-forming processes, finite element (FE) analysis has provided valuable insights into wire rolling. Vallellano et al. [16] performed a 3D numerical analysis using the ABAQUS finite element software to study contact stress distributions, residual stresses, and lateral spread. While 3D FE analysis offers highly detailed information, it is computationally expensive and requires validation against a reliable reference. Whether the

objective is process design or FE validation, a review of the literature further reveals a notable gap in the availability of models with well-defined, traceable assumptions for the rolling of round wire.

Using similar techniques to the asymptotic mathematical modelling of sheet rolling [17], here an asymptotic mathematical model is developed with the assumption of a small friction coefficient and large length-to-thickness ratio, consistent with the slab method of Kazeminezhad and Karimi Taheri [9]. The further assumption of comparably small width is made, which simplifies the problem to plane-stress, allowing for finding the lateral spread without the need for fitting parameters or solving a problem in a complicated 3D stress state. Even so, the derivation of equations in all directions and the use of asymptotic analysis provide a foundation for further improvements to the model in the future, unlike the slab method, which remains inherently limited.

The mathematical model is explained in Section 2. The non-dimensionalised governing equations in all directions are derived in Section 2.1 for a rigid perfectly plastic material, exploiting small parameters of the ratio of half-thickness and width to length of roll gap and the friction coefficient. These assumptions reduce the general deformation state to plane-stress deformation at the leading order, which is solved in Section 2.2. Section 2.3 summarises the solutions and outlines the computational methods used to evaluate them. The experimental procedure for obtaining data is detailed in Section 3, which is used to validate the lateral spread predictions of the model in Section 4.1, alongside FE data from Vallellano et al. [16]. Similarly, the roll pressure predictions are validated against FE data from Vallellano et al. [16] and presented in Section 4.2. Finally, a discussion on potential directions for future research is provided in Section 5.

## 2. Mathematical model

A schematic diagram of the model is shown in figure 2. In the first pass of rolling, the round wire undergoes deformation, resulting in a barrel-like cross-section. With the goal of the simplest possible model a set of assumptions is made to evade the complexity arising from round cross-section at the entrance and barrel-like cross-section afterwards. Firstly, instead of accounting for the circular cross-section at the entrance, it is assumed that at some point after entry, the initially circular cross-section has transformed into a rectangular shape while preserving its area. This stage is chosen as the beginning of the roll gap in the model and is marked A, in figure 2. Therefore, we may write

$$4\hat{h}_A^2 = \frac{\pi}{4} \hat{d}_0^2 \quad \Rightarrow \quad 2\hat{h}_A = \frac{\sqrt{\pi}}{2} \hat{d}_0, \quad (4)$$

where  $\hat{h}_A$  is the half thickness of the wire corresponding to location A. As the quantities will later be non-dimensionalised, and to clearly distinguish between

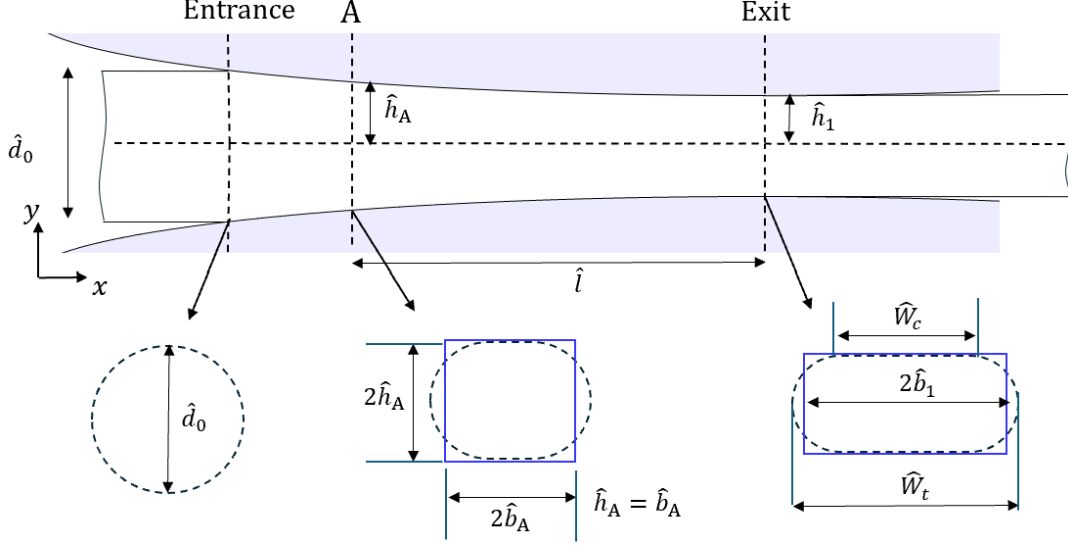


Figure 2: A diagram of the model; the region of interest extends from point A to the roll gap exit. At A, the cross-section is approximated as a square with the same area as the initial round wire, transitioning into a rectangular approximation during rolling with the same area as the real bulged cross section.

dimensional and non-dimensional parameters, we adopt the convention that, for the remainder of the paper, variables with hats denote dimensional quantities, while unhatted variables represent their dimensionless counterparts. Secondly, as the rolling progresses, a square cross-section is assumed to progressively flatten into a rectangular shape, the area of which is assumed to be the same as the area of the real cross-section with bulged edges (see figure 2). Thirdly, the bulged edges are approximated as a half-circle, implying that the bulged radius is taken as half the thickness. This allows the model with a rectangular cross-section to be correlated to the real process. As will be shown later, this set of assumptions allows for accurate lateral spread prediction.

In the following, the governing equations are written for rigid perfectly plastic material rolled with rigid rolls while Coulomb friction is imposed between the rolls and material with a constant friction coefficient.

### 2.1. Scaling and non-dimensionalising the equations

The exploitation of small dimensionless parameters is an important part of the analysis here. We therefore rescale the variables in terms of relevant dimensional scales. The horizontal distance is scaled with the characteristic roll-gap length,  $\hat{\ell}$ , such that  $\hat{x} = \hat{\ell}x$ , and the vertical distance with the initial sheet half-thickness,  $\hat{h}_A$ , so that  $\hat{y} = \hat{h}_Ay$ . For circular rolls of radius  $\hat{R}$ , the roll-gap length  $\hat{\ell}$  is equal to

$$\hat{\ell} = \sqrt{2\hat{R}(\hat{h}_A - \hat{h}_1) - (\hat{h}_A - \hat{h}_1)^2}, \quad (5)$$

where  $\hat{\ell}$  is measured from the point A, as shown in figure 2.

Similar to the slab method, small values of the friction coefficient and the aspect ratio  $\delta = \hat{h}_A/\hat{\ell}$  are of interest.

The latter can be justified by the practical application of small wire diameters (less than a centimetre) and large reductions. For instance, applying a 100 mm roll diameter to an 8 mm wire diameter with a reduction of 30% results in an aspect ratio of  $\delta = 0.37$ , and a larger reduction or a smaller wire diameter would further decrease this value. We also assume that the width,  $\hat{b}(\hat{x})$ , is of the same order of magnitude as the thickness. This is clearly true for the initially cylindrical wire, and becomes less true the wider the wire becomes during rolling. We therefore scale the lateral distance  $\hat{z}$  with the initial sheet half-thickness,  $\hat{h}_A$ , so that  $\hat{z} = \hat{h}_Az$ . In this way, with the assumption of the rectangular cross-section without considering barreling,  $z$  varies between  $-b(x)$  and  $b(x)$  where  $b(x) = \hat{b}(\hat{x})/\hat{h}_A$ .

The velocity  $(\hat{u}, \hat{v}, \hat{w})$ , is scaled with the a priori unknown wire velocity at point A, denoted  $\hat{U}_A$ , and with factors of  $\delta$  to balance incompressibility, giving  $(\hat{u}, \hat{v}, \hat{w}) = (\hat{R}\hat{U}_Au, \delta\hat{R}\hat{U}_Av, \delta\hat{R}\hat{U}_Aw)$ . The actual dimensional sheet velocity  $\hat{U}_A$  may be inferred a posteriori, as the dimensionless surface velocity at the neutral point must correspond to the dimensional roll surface velocity.

Hydrostatic pressure,  $\hat{p}$ , and the normal Cauchy stresses,  $\hat{\sigma}_{xx}$  and  $\hat{\sigma}_{yy}$ , and their deviatoric components,  $\hat{s}_{xx}$  and  $\hat{s}_{yy}$ , are all scaled with the shear yield stress,  $\hat{\kappa}$ . The assumption of a small friction coefficient is encoded by setting  $\mu = \delta\beta$ , where  $\beta$  is a quantity of approximate unit magnitude. Similarly, the shear stress,  $\hat{\sigma}_{xy}$  is scaled with  $\delta\hat{\kappa}$  resulting from small friction. To determine the order of magnitude of stress components in the  $z$ -direction the boundary condition is analysed. The lateral edges of the wire must be tension-free so that the material can move in this direction ( $z$ -direction). This condition may be written as  $\boldsymbol{\sigma} \cdot \mathbf{s} = 0$ , where  $\mathbf{s}$  is the unit vector normal

to the wire lateral edges. For a rectangular cross section,  $\mathbf{s} = (-\delta db/dx, 0, 1)$ . Therefore, at  $z = b(x)$ , we get

$$\delta \frac{db}{dx} \sigma_{xx} = \hat{\sigma}_{xz}/\hat{\kappa}, \quad (6a)$$

$$\delta^2 \frac{db}{dx} \sigma_{xy} = \hat{\sigma}_{yz}/\hat{\kappa}, \quad (6b)$$

$$\delta \frac{db}{dx} \hat{\sigma}_{xz}/\hat{\kappa} = \hat{\sigma}_{zz}/\hat{\kappa}, \quad (6c)$$

where  $db/dx$ ,  $\sigma_{xx}$ , and  $\sigma_{xy}$  are written in their known dimensionless forms, and  $\hat{\sigma}_{zz}$ ,  $\hat{\sigma}_{xz}$ , and  $\hat{\sigma}_{yz}$ , whose orders of magnitude are thus far unknown, are written in dimensional form. The set of equations 6 implies that  $\hat{\sigma}_{xz}$  is  $O(\delta)$  and the normal stress in the lateral direction,  $\hat{\sigma}_{zz}$  and the shear stress component  $\hat{\sigma}_{yz}$  are  $O(\delta^2)$ . Although this holds at the lateral edges, we assume these orders of magnitude apply throughout the modelling region. Consequently,  $\hat{\sigma}_{zz}$  and  $\hat{\sigma}_{yz}$  are scaled with  $\delta^2 \hat{\kappa}$ , while  $\hat{\sigma}_{xz}$  is scaled with  $\delta \hat{\kappa}$ . As will be shown later, this approach leads to the simplest yet consistent solution, where the horizontal velocity and normal stresses depend only on  $x$ . To complete the non-dimensionalisation, the plastic parameter,  $\hat{\lambda}$ , is scaled with  $\hat{U}_A/\hat{\kappa}\hat{\ell}$ .

Under the scaling introduced, the dimensionless form of force balance in each direction is

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0, \quad (7a)$$

$$\delta^2 \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \delta^2 \frac{\partial \sigma_{yz}}{\partial z} = 0, \quad (7b)$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0. \quad (7c)$$

The dimensionless forms of the incompressibility and flow rule relations are

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} + \frac{\partial v}{\partial y} = 0, \quad (8a)$$

$$\frac{\partial u}{\partial x} = \lambda s_{xx}, \quad (8b)$$

$$\frac{\partial v}{\partial y} = \lambda s_{yy}, \quad (8c)$$

$$\frac{\partial w}{\partial z} = \lambda s_{zz}, \quad (8d)$$

$$\frac{\partial u}{\partial y} + \delta^2 \frac{\partial v}{\partial x} = \delta^2 \frac{1}{2} \lambda \sigma_{xy}, \quad (8e)$$

$$\frac{\partial u}{\partial z} + \delta^2 \frac{\partial w}{\partial x} = \delta^2 \frac{1}{2} \lambda \sigma_{xz}, \quad (8f)$$

$$\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \delta^2 \frac{1}{2} \lambda \sigma_{yz}. \quad (8g)$$

where the deviatoric stresses  $s_{ij}$  is defined as

$$\sigma_{ij} = s_{ij} - p \delta_{ij} \quad \text{and} \quad -p = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \delta^2 \sigma_{zz}). \quad (9)$$

From (9) it can be seen that to get a non-zero pressure at leading order,  $s_{zz}$  is also required to be the order of magnitude of  $\hat{\kappa}$ .

With the scaling introduced, the von Mises yield criterion will be

$$(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{xx} - \delta^2 \sigma_{zz})^2 + (\sigma_{yy} - \delta^2 \sigma_{zz})^2 + 6\delta^2 \sigma_{xy}^2 + 6\delta^2 \sigma_{xz}^2 + 6\delta^4 \sigma_{yz}^2 = 6. \quad (10)$$

The Coulomb friction condition on the roll surface  $y = h(x)$  is  $\mathbf{t} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} = \mp \mu \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}$ , where  $\mu = \delta\beta$  is the coefficient of friction,  $\mathbf{n} = (-\delta dh/dx, 1, 0)/\sqrt{1 + \delta^2 (dh/dx)^2}$  is the unit normal to the roll surface and  $\mathbf{t} = (1, \delta dh/dx, 0)/\sqrt{1 + \delta^2 (dh/dx)^2}$  is the unit tangent in the rolling direction. This may be expressed as

$$\delta \frac{dh}{dx} (\sigma_{yy} - \sigma_{xx}) + \left(1 - \delta^2 \left(\frac{dh}{dx}\right)^2\right) \sigma_{xy} = \mp \delta \beta \left( \sigma_{yy} - 2\delta \frac{dh}{dx} \sigma_{xy} + \delta^2 \left(\frac{dh}{dx}\right)^2 \sigma_{xx} \right). \quad (11)$$

Throughout this work, the convention is adopted that the negative sign in  $\mp$  corresponds to the region before the neutral point ( $\hat{x} < \hat{x}_N$ ), while the positive sign applies to the region after the neutral point ( $\hat{x} > \hat{x}_N$ ).

Assuming dimensional tensions  $\hat{F}_{A/out}$  are applied at location A and at the exit, the horizontal stress must satisfy

$$\frac{\hat{F}_{A/out}}{4\hat{\kappa}\hat{h}_A^2} = \int_{-b_{A/out}}^{b_{A/out}} \int_{-h_{A/out}}^{h_{A/out}} \sigma_{xx} dy dz, \quad (12)$$

where  $h_{out}$  and  $b_{out}$  are half of the final thickness and width respectively. Vertical symmetry requires the boundary conditions

$$\sigma_{xy}(x, 0, z) = 0 \quad \text{and} \quad v(x, 0, z) = 0. \quad (13)$$

The no-flux constraints on the roll surface  $y = h(x)$ , and on the wire edge  $z = b(x)$ , lead respectively to

$$v = \delta \frac{dh}{dx} u, \quad w = \delta \frac{db}{dx} u. \quad (14)$$

Finally, Integrating the mass conservation law in (8a) from  $y = -h(x)$  to  $y = h(x)$  and  $z = -b(x)$  to  $z = b(x)$  yields

$$\int_{-b(z)}^{b(z)} \int_{-h(z)}^{h(z)} u dy dz = 4. \quad (15)$$

## 2.2. Leading-order asymptotic solution

We now proceed by expanding all variables in an asymptotic series in  $\delta$ , so that, for example,  $\sigma_{xx} = \sigma_{xx}^{(0)} + \delta \sigma_{xx}^{(1)} + \delta^2 \sigma_{xx}^{(2)} + \dots$ , and collecting like terms. While this formalism could be used to compute the solution to an arbitrary order of accuracy (for example, Erfanian et al. [17] consider up to the second order terms  $\sigma_{xx}^{(2)}$  for sheet rolling), here we will only consider the leading order terms.

We therefore neglect all terms of  $O(\delta)$  or higher to derive the leading-order equations. Substituting the hydrostatic pressure (9) into the yield function (10) at leading order gives

$$\left(\sigma_{xx}^{(0)} - (-3p^{(0)} - \sigma_{xx}^{(0)})\right)^2 + \left(\sigma_{xx}^{(0)}\right)^2 + \left(3p^{(0)} + \sigma_{xx}^{(0)}\right)^2 = 6 \quad (16a)$$

$$\Rightarrow \sigma_{xx}^{(0)} = -\frac{3p^{(0)}}{2} + \frac{\sqrt{4 - 3p^{(0)2}}}{2}, \quad (16b)$$

$$\sigma_{yy}^{(0)} = -\frac{3p^{(0)}}{2} - \frac{\sqrt{4 - 3p^{(0)2}}}{2}. \quad (16c)$$

The force balance in the  $y$  direction (7b), at leading order is reduced to

$$\frac{\partial \sigma_{yy}^{(0)}}{\partial y} = 0. \quad (17)$$

Using this condition alongside the stress solutions (16) implies that  $\sigma_{yy}^{(0)}$  and therefore  $p^{(0)}$  and  $\sigma_{xx}^{(0)}$  are vertically homogeneous which is the same as the slab method's assumption. We now show that these components are independent of  $z$ , as well, and only change along the rolling direction,  $x$ . From (8e) and (8f),  $u^{(0)}$  is independent of  $y$  and  $z$ , which, together with the continuity equation (8a) and (8g), imply that  $v^{(0)}$  is linear in  $y$  and  $w^{(0)}$  is linear in  $z$ . Now the set of equations (8b)–(8d), where their left-hand sides are only functions of  $x$  while the stresses are functions of  $p^{(0)}$  (equation (16)) shows that in order for a single value for  $\lambda^{(0)}$  satisfies them all,  $p^{(0)}$  must be a function of  $x$  only.

The force equilibrium in the  $x$  direction (7a) now can be integrated over  $z$  and  $y$  to give

$$h(x)b(x)\sigma_{xx}^{(0)}(x) + b(x)\sigma_{xy}^{(0)}(x, h(x)) + h(x)\sigma_{xz}^{(0)}(x, b(x)) = 0, \quad (18)$$

where  $\sigma_{xy}^{(0)}(x, h(x))$  and  $\sigma_{xz}^{(0)}(x, b(x))$  can be found from the boundary conditions (11) and (6a), respectively. Therefore

$$hb \frac{d\sigma_{xx}^{(0)}}{dx} + \sigma_{xx}^{(0)} \left( b \frac{dh}{dx} + h \frac{db}{dx} \right) + b \left( -\frac{dh}{dx} \mp \beta \right) \sigma_{yy}^{(0)} = 0. \quad (19)$$

This is the same equation as was derived using the slab method by Kazeminezhad and Karimi Taheri [9], although our formulae for  $\sigma_{xx}^{(0)}$  and  $\sigma_{yy}^{(0)}$  here are different. Equation (19) provides an ordinary differential equation (ODE) in terms of  $p^{(0)}$  and  $b(x)$ , when combined with stress solutions (16).

Another ODE results from the velocity equations. Equations (8e) and (8f) at leading order show that  $u^{(0)}$  has no  $y$  or  $z$  dependence and so is only a function of  $x$ . The average mass balance (15) then requires that

$$u^{(0)}(x) = \frac{1}{b(x)h(x)}. \quad (20)$$

From flow rules (8b)–(8d),  $\lambda^{(0)}$  is a function of  $x$ . Therefore, from the tension flow rule in the  $y$  direction (8c)

along with the symmetry condition (13), the vertical velocity is found as

$$v^{(0)} = y\lambda^{(0)}s_{yy}^{(0)}. \quad (21)$$

Solutions (20) and (21), along with the no-penetration surface condition (14), results in

$$\lambda^{(0)} = \frac{dh/dx}{bh^2s_{yy}^{(0)}}, \quad (22)$$

which holds not only on the surface but throughout the entire thickness, given that  $\lambda^{(0)}$  is only a function of  $x$ . Similarly, from the tension flow rule in the  $z$  direction (8d) coupled with no-flow normal to the edges condition (14),

$$\lambda^{(0)} = \frac{db/dx}{b^2hp^{(0)}}. \quad (23)$$

Matching (22) and (23) gives an ODE for  $b(x)$  as

$$\frac{db}{dx} = \frac{b}{h} \frac{dh}{dx} \frac{p^{(0)}}{s_{yy}^{(0)}}. \quad (24)$$

By solving equations (24) together with (19),  $b(x)$  and  $p^{(0)}$  are solved. However, in equation (19) the dependency on  $b(x)$  can be removed to simplify further; an alternative ODE for  $b(x)$  may be found by replacing  $u^{(0)}$  from (20) and  $\lambda^{(0)}$  from (23) into (8b), as

$$b \frac{dh}{dx} + h \frac{db}{dx} = -h \frac{db}{dx} \frac{s_{xx}^{(0)}}{p^{(0)}}. \quad (25)$$

Replacing expression (25) into (19) and further simplifying the resultant using (24) gives

$$\frac{d\sigma_{xx}^{(0)}}{dx} - \left( \frac{dh/dx}{h} \frac{s_{xx}^{(0)}}{s_{yy}^{(0)}} \right) \sigma_{xx}^{(0)} + \frac{1}{h} \left( -\frac{dh}{dx} \mp \beta \right) \sigma_{yy}^{(0)} = 0, \quad (26)$$

which is an ODE for the stresses without depending on the width. From (12), the boundary conditions at leading order at point A and the roll-gap exit are

$$4\sigma_{xx}^{(0)}(x = A) = F_A \quad \text{and} \quad 4bh\sigma_{xx}^{(0)} \Big|_{x=1} = F_{\text{out}}. \quad (27)$$

With this, equation (26) can be solved from point A forward with  $-ve$  sign and from the exit backwards with  $+$  sign. It is interesting to note from equations (24) and (19) that  $b(x)$ , and consequently the width of contact,  $W_c$ , and the lateral spread of the wire,  $W_L$ , depend only on the initial diameter of the wire, reduction ratio, roll radius, and friction coefficient, and not on the roll speed on the material type (e.g. the yield stress  $\hat{\kappa}$ ). The influence of roll speed and material type would become apparent if hardening were incorporated into the analysis, although they likely remain negligible in comparison.



### 2.3. Summary and numerical evaluation

In this section, a simple numerical procedure for performing the calculations is detailed.

Equation (26) can be written in terms of  $p^{(0)}$  by using

$$\frac{d\sigma_{xx}^{(0)}}{dx} = \frac{d\sigma_{xx}^{(0)}}{dp^{(0)}} \frac{dp^{(0)}}{dx}, \quad (28)$$

and replacing stress components from (16) and (9). As described by equation (27), the force at point A is necessary for solving the ODE (26). However, it is assumed that force (and as a result pressure) at point A is the same as that at the roll-gap entrance. This assumption will be justified in the following section using FE data, which shows a pressure drop following the large peak at the entrance. The exit point in the model is the same as the roll-gap exit, therefore,  $F_{\text{out}}$  determines the pressure at the exit. If there is no exit tension, then  $\sigma_{xx}^{(0)}$  becomes zero at the exit. Otherwise, the value of  $\sigma_{xx}^{(0)}$  at the exit depends on the final width. In such a case,  $b$  at the exit must be estimated and iteratively refined to align with the width determined from (24) or (25).

The solution for  $p^{(0)}$  is chosen to satisfy the forward and backward tension conditions, which are taken to be zero for the results presented below. Therefore,  $p^{(0)}$  is solved by integrating equation (26) forward from the entrance with positive sign of friction, and integrating equation (26) backwards from the exit with positive sign of friction, using the MATLAB ODE solver `ode45` [18]. This is the same solution as the slab method, and the two curves thus produced are referred to as the pressure hill, and the intersection determines the location of the neutral point.

After solving for stresses,  $b(x)$  and consequently the lateral spread,  $W_t$ , is determined by integrating either equation (24) or (25) from point A to the exit. The integration is performed using the MATLAB ODE solver `ode45`, with the initial condition  $b(x = A) = 1$ , which is consistent with the chosen non-dimensionalisation. As detailed in the introduction, the area of the rectangular cross-section is assumed to be equal to that of the actual wire in the roll gap. As a result, the lateral spread is determined by equating the cross-sectional area of the real shape to that of the rectangle. Therefore

$$2\hat{W}_c\hat{h} + \pi\hat{h}^2 = 4\hat{b}\hat{h} \quad \Rightarrow \quad \hat{W}_c = 2\hat{b} - \frac{\pi\hat{h}}{2}, \quad (29)$$

where  $\hat{W}_c$  is the width in contact with the roll, and the bulged edges are assumed to form a half-circle. With this, the lateral spread,  $\hat{W}_t$ , is found as

$$\hat{W}_t = \hat{W}_c + 2\hat{h} \quad (30)$$

The total computation time varies depending on the tension at the exit, and is at most in the order of seconds on a standard laptop. For zero tension, solving stresses and lateral spread take less than a second.

### 3. Experimental Methodology

Experimental verification was carried out using a Hille 25 rolling mill using D2 tool steel rollers with a roll diameter of 100 mm. The mill operates at a typical speed of 60 rpm. Stainless steel 316 rods of initial diameters 2.96, 3.96, 5.96 and 7.96 mm were rolled without lubrication at room temperature to reductions of 30–60% in a single pass. The initial and final lengths, thicknesses and widths were all measured using a Mitutoyo 293-240-30 micrometer.

### 4. Results and discussion

#### 4.1. Lateral spread

The lateral spread prediction,  $\hat{W}_t$ , for both cases, is plotted in Figure 3 for various diameters and reduction ratios with 50 mm roll radius. The results are compared with experimental data, Kobayashi's empirical equation (3), and empirical equation (2) due to Kazeminezhad and Karimi Taheri [8]. The latter equation is particularly relevant because the parameters in (2) were fitted for stainless steel. In the current model, predictions depend on the friction coefficient, which is challenging to determine experimentally. The experimental data were obtained under non-lubricated test conditions, and a friction coefficient of  $\mu = 0.25$  has been shown to provide the best agreement with experimental data. Consequently, this value is used for results in Figure 3. The current model fails at larger reductions for  $\hat{d}_0 = 2.96$  mm and  $\hat{d}_0 = 3.96$  mm, as indicated by the unfinished lines, due to a singularity in pressure (see equation 16); interestingly, the experiments also encountered difficulties achieving comparable larger reductions during single-pass rolling in these cases, and so experimental data in these cases is also lacking. Where data is available, the model agrees closely with the experimental data for all diameters and reduction ratios. Kazeminezhad Equation (2) depends only on the reduction ratio. While it performs well for smaller reduction ratios, it deviates for larger values, with the deviation appearing to depend on wire diameter (e.g., underestimates for  $\hat{d}_0 = 5.96$  mm and overestimates for  $\hat{d}_0 = 7.96$  mm). Kobayashi's equation depends on roll radius and wire diameter as well as the reduction ratio and seems to be derived for small wire diameter conditions. It underestimates the lateral spread across all wire diameters, particularly at higher reduction ratios, and is the least reliable for predicting the set of experimental data presented in Figure 3.

Some studies suggest that the friction coefficient has a negligible effect on lateral spread [2, 10], attributing this to the movement of lubricant toward the roll edges, driven by the extremely high contact pressure at the entry point. Similarly, the empirical equations (2) and (3) exhibit no dependence on friction. To examine the impact of friction, the ratio of lateral spread to the initial wire diameter is

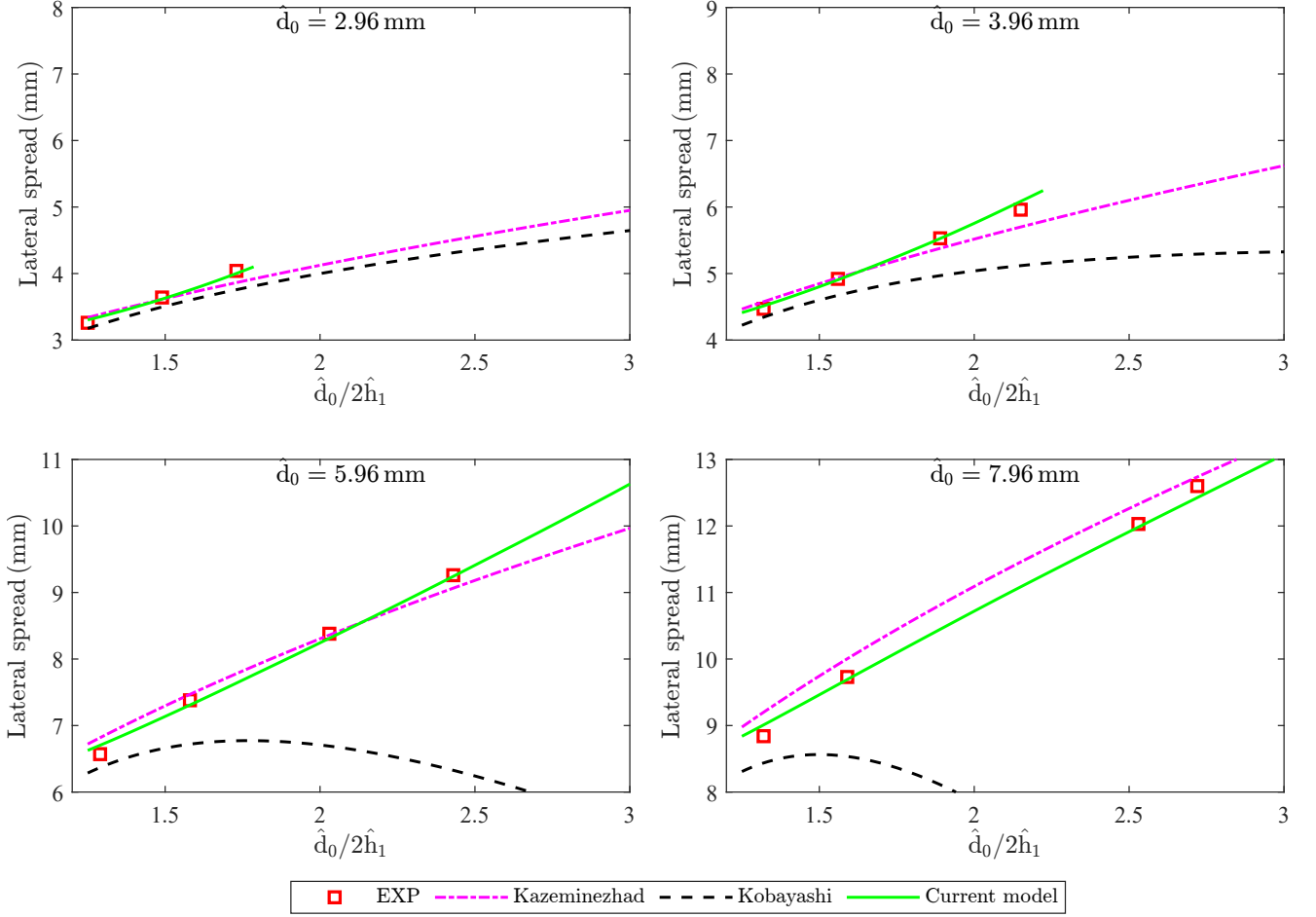


Figure 3: Lateral spread for stainless steel wires with different diameters,  $\hat{d}_0$ , and reduction ratios,  $\hat{d}_0/(2\hat{h}_1)$  from different methods; experimental data, empirical equation 2 from Kazeminezhad and Karimi Taheri [8], empirical equation from Kobayashi (3), and the current model. Other parameters used are  $(\hat{R}, \mu) = (50 \text{ mm}, 0.25)$ .

plotted in Figure 4 for two different friction coefficients. In the absence of experimental data under the lubricated condition, the FE results from Vallengano et al. [16] are utilised. The data reported in [16] corresponds to a wire with a 5 mm diameter, rolled using 75 mm radius rolls, with no applied forward or backward tension. The yield stress,  $\hat{Y}$ , is assumed to be constant and equal to 385 MPa the same as the average value in FE simulations. The Tresca friction law,  $\hat{\tau} = m\hat{\kappa}$  with the coefficient of  $m = 0.25$  is used to generate FE results in [16]. Generally, the relationship  $\mu < m/\sqrt{3}$  is used essentially for a simple elastoplastic model [19], where the upper limit,  $\mu = 0.14$ , is used to transfer  $m$  into the Coulomb friction coefficient,  $\mu$ . The data represented by the black line with circles corresponds to the experimental measurements for  $\hat{d}_0 = 5.96 \text{ mm}$  without lubrication ( $\mu = 0.25$ ), while the counterpart for  $\hat{d}_0 = 5 \text{ mm}$  is derived from FE simulations using  $m = 0.25$  (approximated by  $\mu = 0.14$ ). To minimise the influence of different wire diameters in this comparison, wires with relatively similar diameters were selected, and the results were scaled by the initial diameter.

The comparison in Figure 4 indicates that friction

does influence lateral spread, particularly at higher reductions. Greater friction appears to increase lateral spread. This can be explained by the increased constraint in the longitudinal direction. Although friction acts in both the lateral and rolling directions, the lateral contact length is smaller than the longitudinal contact length. As a result, friction primarily acts as a resistance in the rolling direction. This effect is accurately captured by the current model. For this wire diameter, Kazeminezhad's equation correctly predicts lateral spread for larger friction coefficients but tends to overestimate the results for lower friction values. Conversely, Kobayashi's equation demonstrates better agreement under low-friction conditions compared to high friction, although different roll and wire diameters might also influence this.

#### 4.2. Roll pressure

The roll pressure  $\sigma_{yy}^{(0)}$  is found from (16) and plotted with respect to the contact angle,  $\phi$  in Figure 5 for different reduction ratios. Results are compared with FE simulations from Vallengano et al. [16], the slab model

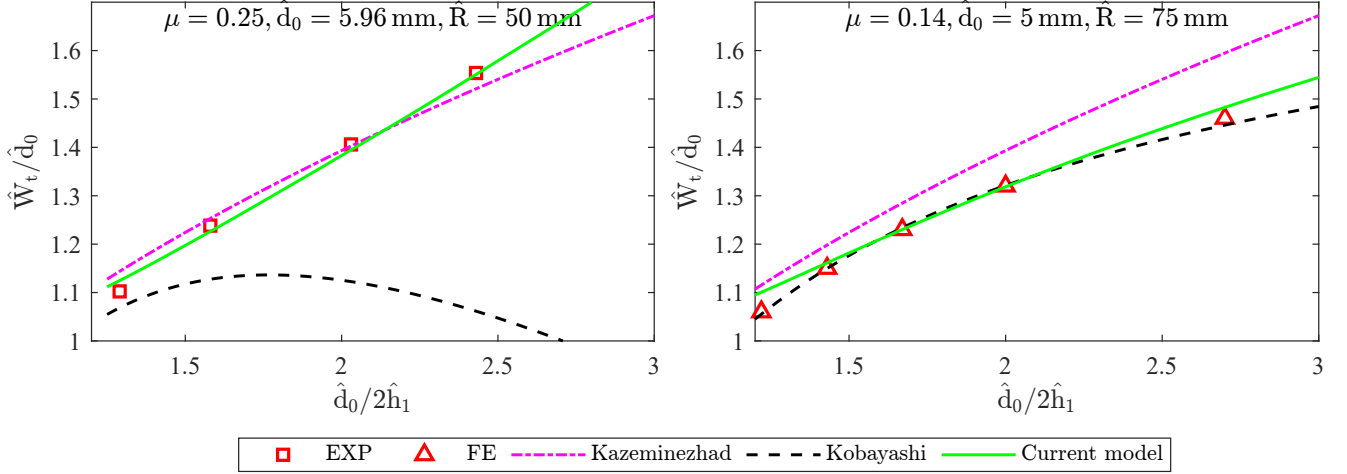


Figure 4: Effect of friction coefficient on lateral spread. Results are plotted from different methods; experimental data, FE simulations [16], empirical equation 2 from Kazeminezhad and Karimi Taheri [8], empirical equation from Kobayashi (3), and the current model.

from Kazeminezhad and Karimi Taheri [9] and the plane-strain slab model for the rolling of thin sheet in Erfanian et al. [17]. The same parameters as those used in Valletano et al. [16] and described in the previous section are applied. The results for plane stress and plane strain are calculated from point A in Figure 2.

FE results for all reductions can be seen to have two distinct regions; a massive rise between the entrance and point A and a shallow rise from point A to the exit. This trend can be explained better when looking alongside Figure 6 which schematically shows how the wire looks as seen from the side and above while rolling. The wire enters between rolls from the right-hand side with a circular cross-section and exits from the left with an almost rectangle cross-section, therefore, the contact surface forms rather like half of an ellipse. As explained by Carlsson [2] when the material starts to deform, surrounding parts of the wire are still in the elastic range and will therefore resist deformation. The situation may be compared to that of an indentation. As a result, a normal pressure will build up, resulting in a sharp rise in contact pressure soon after the entrance. As deformation continues, larger parts of the wire deform, reducing resistance from the remaining elastic regions, which in turn results in a pressure drop until point A. At this point, the material begins to flow laterally, and a typical friction hill develops, with a pressure peak forming between this point and the roll exit [2].

From the results, it can be seen that the current model correctly predicts the location of point A by assuming that the area of the cross-section at this point is the same as that of the wire before rolling. Both the current plane-stress model and Kazeminezhad and Karimi Taheri [9] correctly predict the roll pressure at point A, for larger reductions. From point A to the roll exit, the FE results lie between the plane-strain and plane-stress predictions, reflecting the presence of 3D material flow

within the roll gap. Consequently, the results presented by Kazeminezhad and Karimi Taheri [9] show better agreement with the FE data. Another notable difference between the FE results and the perfect plastic models in Figure 5 is that the roll pressure starts at zero at the entrance and returns to zero at the exit. This behaviour arises from the inclusion of elasticity in the FE simulations, which is absent in the perfect plastic models.

## 5. Conclusion

This work develops a mathematical framework for wire rolling, emphasising the accurate prediction of lateral spread. The modeling of a rectangular cross-section is correlated to the real process by focusing on the region of the roll gap where lateral flow begins and assuming that, beyond this point, the cross-section in the real process can be approximated by a rectangular shape in the model with a matching area. Also, a general state of deformation is simplified to the plane-stress deformation with the assumption of a small width with respect to the length. The set of assumptions made leads to the simplest possible first-principles model, which is shown to successfully predict lateral spread across a range of wire diameters, reduction ratios, and different friction coefficients. Importantly, this model achieves accurate predictions without requiring empirical fitting parameters or assuming additional factors. This holds true even though the assumption of plane-stress deformation is not strictly satisfied; comparison of roll pressure predictions with FE results suggests that the actual pressure distribution falls between that of plane-stress and plane-strain deformation. A possible explanation for this lies in the importance of the stress ratio,  $s_{zz}/s_{yy}$ , rather than the absolute stress values, in predicting the lateral spread; while the model may underestimate individual stresses, it appears to capture the correct ratio, leading to accurate predictions of lateral spread.



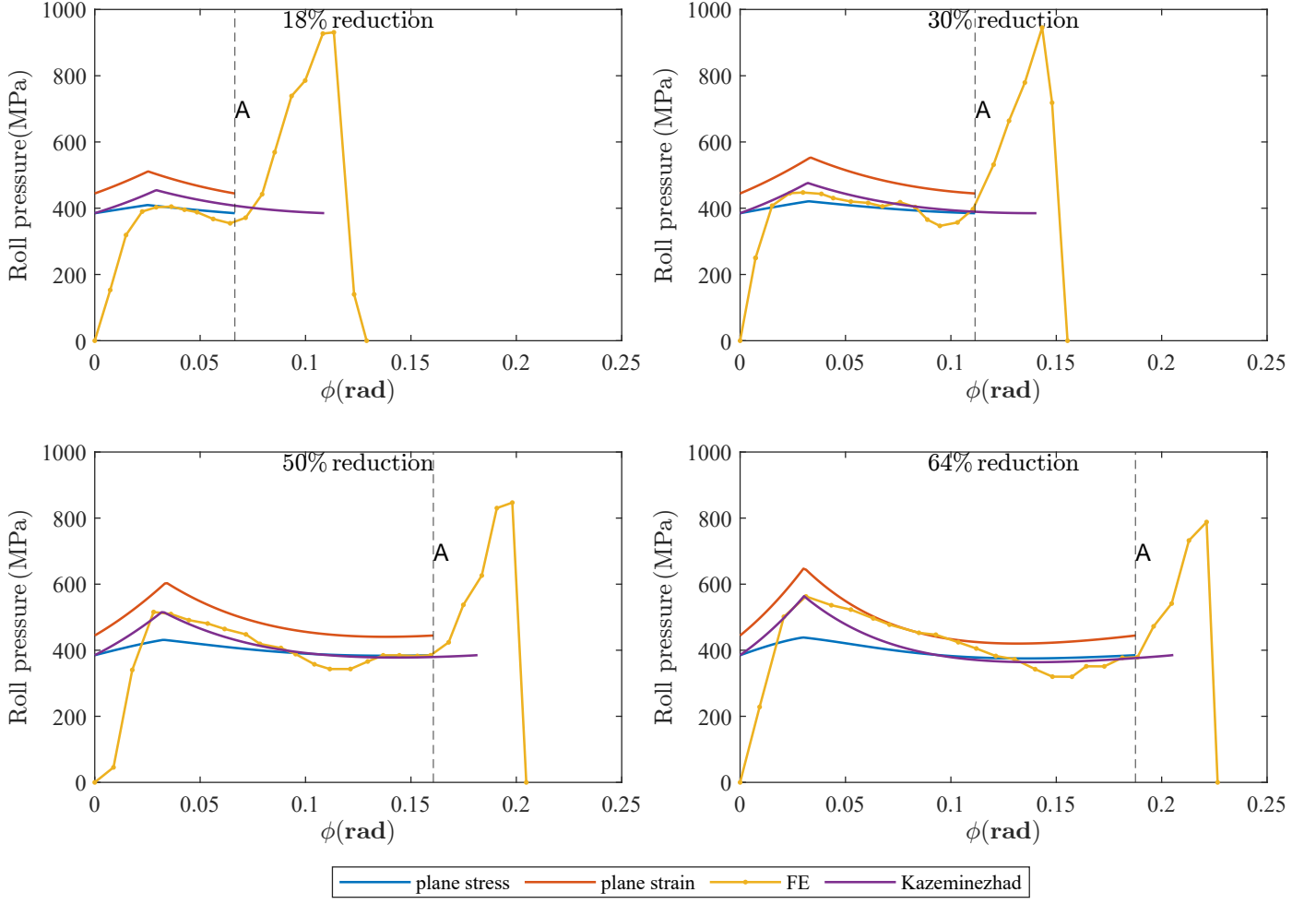


Figure 5: Roll pressure from different methods; plane-strain slab method [17], current plane-stress method, and FE simulations [16] and Kazeminezhad and Karimi Taheri [9] slab method. Plotted for different reduction ratios against contact angle  $\phi$ . Other parameters used are  $(\bar{\kappa}, \hat{d}_0, \hat{R}, \mu) = (385/\sqrt{3} \text{ MPa}, 5 \text{ mm}, 75 \text{ mm}, 0.14)$ .

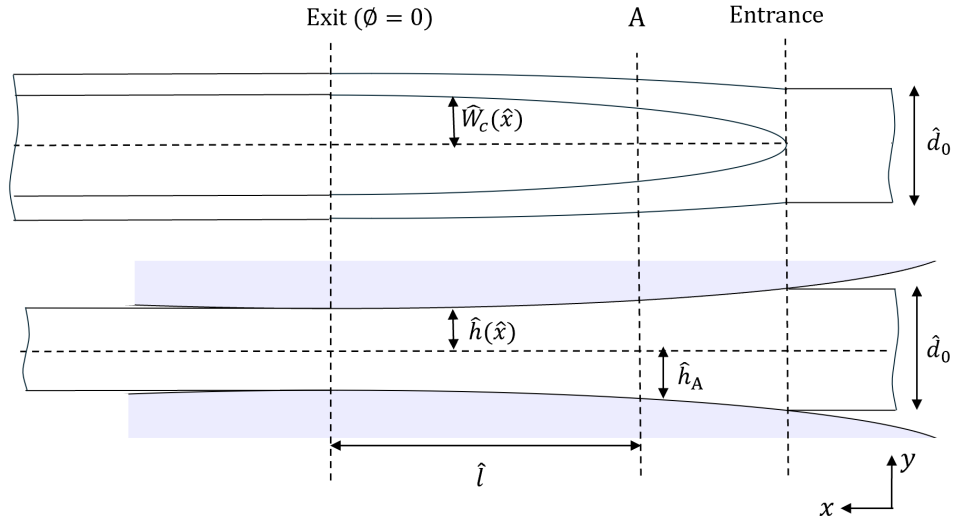


Figure 6: A diagram of the deforming wire, seen from above (top figure) and from side (bottom figure). For symbols, refer to the main text. Adapted from Figure 3 of Carlsson [2].

Nonetheless, having formulated stress and strain equations in all directions lays the foundation for a more general model. 3D numerical studies conducted using ABAQUS software by Carlsson [2] and Vallellano et al. [16] show that the contact pressure distribution is more complex than the traditional pressure hill profile. Relaxing the assumption of a small rate of change in width in the current model would allow the problem to be modelled under a general stress state. Also, solving the higher order terms would enable accounting for the curvature of the edges. Additionally, further studies are required to explore potential correlations between the challenges faced in experiments when achieving certain reductions in a single pass and the pressure singularity predicted by the current model.

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## References

- [1] H. Utsunomiya, P. Hartley, I. Pillinger, Three-dimensional elastic-plastic finite-element analysis of the flattening of wire between plain rolls, *J. Manuf. Sci. Eng.* 123 (2001) 397–404. doi:[10.1115/1.1365158](https://doi.org/10.1115/1.1365158).
- [2] B. Carlsson, The contact pressure distribution in flat rolling of wire, *J. Mater. Process. Technol.* 73 (1998) 1–6. doi:[10.1016/S0924-0136\(97\)00091-5](https://doi.org/10.1016/S0924-0136(97)00091-5).
- [3] N. R. Chitkara, W. Johnson, Some experimental results concerning spread in the rolling of lead, *J. Basic Eng.* 88 (1966) 489–499. doi:[10.1115/1.3645884](https://doi.org/10.1115/1.3645884).
- [4] G. D. Lahoti, S. Kobayashi, On Hill's general method of analysis for metal-working processes, *Int. J. Mech. Sci.* 16 (1974) 521–540. doi:[10.1016/0020-7403\(74\)90018-6](https://doi.org/10.1016/0020-7403(74)90018-6).
- [5] S. I. Oh, S. Kobayashi, An approximate method for a three-dimensional analysis of rolling, *Int. J. Mech. Sci.* 17 (1975) 293–305. doi:[10.1016/0020-7403\(75\)90010-7](https://doi.org/10.1016/0020-7403(75)90010-7).
- [6] A. E. G. El-Nikhaily, Metal Flow Models for Shape Rolling, Ph.D. thesis, Technical University of Aachen, 1979.
- [7] K. F. Kennedy, A method for analyzing spread, elongation and bulge in flat rolling, *J. Eng. Ind.* 109 (1987) 248–256. doi:[10.1115/1.3187126](https://doi.org/10.1115/1.3187126).
- [8] M. Kazeminezhad, A. Karimi Taheri, A theoretical and experimental investigation on wire flat rolling process using deformation pattern, *Materials & Design* 26 (2005) 99–103. doi:[10.1016/j.matdes.2004.06.010](https://doi.org/10.1016/j.matdes.2004.06.010).
- [9] M. Kazeminezhad, A. Karimi Taheri, Calculation of the rolling pressure distribution and force in wire flat rolling process, *J. Mater. Process. Technol.* 171 (2006) 253–258. doi:[10.1016/j.jmatprotec.2005.06.070](https://doi.org/10.1016/j.jmatprotec.2005.06.070).
- [10] M. Kazeminezhad, A. Karimi Taheri, A. Kiet Tieu, A study on the cross-sectional profile of flat rolled wire, *J. Mater. Process. Technol.* 200 (2008) 325–330. doi:[10.1016/j.jmatprotec.2007.09.029](https://doi.org/10.1016/j.jmatprotec.2007.09.029).
- [11] S. L. Semiatin, J. J. Jonas, Formability and workability of metals: plastic instability and flow localization, *Am. Soc. Metals*, 1984.
- [12] A. Pesin, V. Salganik, E. Trahtengertz, M. Cherniahovsky, V. Rudakov, Mathematical modelling of the stress-strain state in asymmetric flattening of metal band, *J. Mater. Process. Technol.* 125 (2002) 689–694. doi:[10.1016/S0924-0136\(02\)00353-9](https://doi.org/10.1016/S0924-0136(02)00353-9).
- [13] R. Hill, Private communication to B.I.S.R.A., 1955. Letter to A. W. McCrum.
- [14] Applied Mechanics Group, L. G. M. Sparling, Formula for 'spread' in hot flat rolling, *Proc. Inst. Mech. Eng.* 175 (1961) 604–640. doi:[10.1243/PIME\\_PROC\\_1961\\_175\\_043\\_02](https://doi.org/10.1243/PIME_PROC_1961_175_043_02).
- [15] M. Kobayashi, Influence of rolling conditions on spreading in flat rolling of round wire, *J. Jpn Soc. Technol. Plast.* 19 (1978) 630–637. (in Japanese).
- [16] C. Vallellano, P. A. Cabanillas, F. J. Garcia-Lomas, Analysis of deformations and stresses in flat rolling of wire, *J. Mater. Process. Technol.* 195 (2008) 63–71. doi:[10.1016/j.jmatprotec.2007.04.124](https://doi.org/10.1016/j.jmatprotec.2007.04.124).
- [17] M. Erfanian, E. J. Brambley, F. Flanagan, D. O'Kiely, A. N. O'Connor, Through-thickness modelling of metal rolling using multiple-scale asymptotics, 2025. [arXiv:2408.01347](https://arxiv.org/abs/2408.01347), *eur. J. Mech. A/Solids*. (accepted May 2025).
- [18] MathWorks Inc., MATLAB version: 24.1.0.2568132 (R2024a) Update 1, 2024. URL: <https://www.mathworks.com>.
- [19] D.-W. Zhang, H. Ou, Relationship between friction parameters in a Coulomb–Tresca friction model for bulk metal forming, *Tribology International* 95 (2016) 13–18. doi:[10.1016/j.triboint.2015.10.030](https://doi.org/10.1016/j.triboint.2015.10.030).