A well-posed boundary condition for acoustic liners in straight ducts with flow*

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The Myers boundary condition for acoustics within flow over an acoustic lining has been shown to be illposed, leading to numerical stability issues in the time domain and mathematical problems with stability analyses. This paper gives a modification (for flat or cylindrical straight ducts) to make the Myers boundary condition well-posed, and indeed more accurate, by accounting for a thin inviscid boundary layer over the lining, and correctly deriving the boundary condition to first order in the boundary-layer thickness. The modification involves two integral terms over the boundary layer. The first may be written in terms of the mass, momentum, and kinetic energy thicknesses of the boundary layer, which are shown to physically correspond a modified boundary mass, modified grazing velocity, and a tension along the boundary. The second integral term is related to the critical layer within the boundary layer. A time domain version of the new boundary condition is proposed, although not implemented.

The modified boundary condition is validated against high-fidelity numerical solutions of the Pridmore-Brown equation for sheared inviscid flow in a cylinder. Absolute instability boundaries are given for certain examples, though convective instabilities appear to always be present at certain frequencies for any boundary layer thickness.

I. Introduction

Modern turbofan aeroengines almost universally use acoustic linings within the intake and bypass ducts to reduce noise. Simple mathematical models in simple geometries and more complicated numerical simulations of more realistic geometries are used to optimize the type and placement of such linings in order to obtain the greatest benefit. Commonly, the interaction of acoustics with an acoustic lining is specified using the impedance of the lining. Taking the fluctuating pressure within the fluid to be $p(\boldsymbol{x})\exp\{\mathrm{i}\omega t\}$ and the velocity to be $\boldsymbol{v}(\boldsymbol{x})\exp\{\mathrm{i}\omega t\}$, the impedance of the lining is $Z = p/(\boldsymbol{v}\cdot\boldsymbol{n})$, where $\boldsymbol{n}$ is the surface normal pointing out of the fluid (Z will usually be a function of frequency ω). If the acoustics is on top of a mean flow that slips across the lining, it is well known that this boundary condition must be modified [1–7]. If the fluid velocity at the boundary is $\boldsymbol{U} + \boldsymbol{u}\exp\{\mathrm{i}\omega t\}$, then the boundary condition becomes

$$\mathrm{i}\omega\boldsymbol{u}\cdot\boldsymbol{n} = \left(\mathrm{i}\omega + \boldsymbol{U}\cdot\boldsymbol{\nabla} - (\boldsymbol{n}\cdot\boldsymbol{\nabla}\boldsymbol{U})\cdot\boldsymbol{n}\right)p/Z. \tag{1}$$

This follows from matching fluid and solid normal displacement, rather than normal velocity, and is known as the Myers [7] or Ingard–Myers [4, 7] boundary condition, although this equation (apart from the final term) was earlier given by, for example, Miles [1, equation (3.3′)]. This, in straight ducts for which the final term is identically zero, was shown to be the correct asymptotic limit of a vanishingly-thin inviscid boundary layer independently by Eversman & Beckemeyer [5] and Tester [6], though the boundary needed to be extremely thin to attain this limit in some cases [6, 8]. For numerical simulations, this condition therefore provides an attractive alternative to resolving the thin boundary layer over the lining.

The final term in eq. (1) is nonzero for complicated geometries and flows for which the flow just outside the boundary layer is non-parallel [e.g. 9, 10]. The term is necessary for a *compliant* oscillating boundary, as the boundary motion causes the boundary to see a different normal mean flow velocity at different parts of its oscillation. However, the validity of this assumption for a fixed *permeable* acoustic liner is questionable [see, e.g. 11], with the proof that eq. (1) is the correct boundary condition for a vanishingly-thin inviscid boundary layer only having been given for straight ducts [5, 6]. As in those papers, in this paper we will only consider straight ducts, for which this Myers non-parallel flow term is identically zero.

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Unfortunately, eq. (1) applied with slipping flow leads to numerical instabilities [see, for example, 9, 12] and mathematical illposedness [13]. In order to circumvent slipping flow, there has been recent interest in acoustics within sheared mean flow [14, 15], following the foundations set by Pridmore-Brown [16] and Mungur & Plumblee [17]. This paper will concentrate on the effect of a thin but finite-thickness inviscid boundary layer over the acoustic lining by incorporating the first- and second-order corrections in the boundary layer thickness. Such a situation has been investigated before [6, 8, 18–27], at least to first order; in particular, the derivation in the first part of §III is similar to the derivation of [24], as detailed in §III, although [24] do not allow for temperature or density variations within their boundary layer, nor cover the material presented in the remainder of this paper. To the authors knowledge, the present paper is the first time that a simple asymptotically-valid Myers-type boundary condition incorporating a finite-thickness compressible boundary layer has been proposed, compared to high-fidelity numerical simulations of the Pridmore-Brown equation, and had a stability analysis performed.

Recently, and concurrently with the current work, Rienstra & Darau [28, 29] proposed a modified Myers boundary condition based on a two-dimensional incompressible model. Their results are compared to those derived here in §VI.

II. Governing equations

Our governing equations are the equations of motion for an inviscid compressible perfect gas [30],

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0, \qquad \rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}p, \qquad \frac{\mathrm{D}p}{\mathrm{D}t} = \frac{\gamma p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}, \tag{2}$$

where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$, ρ is the density, $\boldsymbol{u}$ is the velocity, p is the pressure, and γ is the ratio of specific heats.

Although what follows is equally valid for a rectangular duct, we will concern ourselves here with a cylindrical duct, described by (x, r, θ) coordinates with the x-axis running along the centreline of the cylinder. We take the flow to be (for real integer m)

$$\boldsymbol{u} = U(r)\boldsymbol{e_x} + \big(\tilde{u}(r)\boldsymbol{e_x} + \tilde{v}(r)\boldsymbol{e_r} + \tilde{w}(r)\boldsymbol{e_\theta}\big)\exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\}, \tag{3a}$$

$$p = P + \tilde{p}(r)\exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\}, \tag{3b}$$

$$\rho = R(r) + \tilde{\rho}(r)\exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\}, \tag{3c}$$

where the tilde denotes acoustic quantities which are assumed to be small compared with the steady mean flow. We nondimensionalize distance by the radius of the cylinder, velocity by the centreline mean-flow speed of sound $C_0 = \sqrt{\gamma P/R(0)}$, and density by the centreline mean-flow density $R(0)$, implying that pressure is nondimensionalized by γP where P is the mean-flow pressure. Under this nondimensionalization, the duct wall is at $r = 1$, $R(0) = 1$, $P = 1/\gamma$, $M = U(0)$ is the centreline mean-flow Mach number, and ω is the Helmholtz number.

Note that $U(r)$ and $R(r)$ may be arbitrary functions of r as far as this paper is concerned. In reality, gradients of $U(r)$ are caused by viscosity and gradients of $R(r)$ (since $R \propto P/T$, where T is the temperature) are caused by thermal conduction and viscous dissipation (see, e.g. [11] for further details). These mechanisms are here considered insignificant for the acoustic perturbations, which is the standard acoustic assumption, and are considered to have been already accounted for in the mean flow through non-constant $U(r)$ and $R(r)$.

Substituting eq. (3a–c) into eq. (2) and taking only terms linear in the acoustic quantities gives the linearized mass, momentum and energy equations as

$$\mathrm{i}(\omega - Uk)\tilde{\rho} = -\tilde{v}\frac{\mathrm{d}R}{\mathrm{d}r} - \frac{R}{r}\frac{\mathrm{d}(r\tilde{v})}{\mathrm{d}r} + \frac{\mathrm{i}}{r}mR\tilde{w} + \mathrm{i}kR\tilde{u},$$

$$\mathrm{i}R(\omega - Uk)\tilde{v} = -\frac{\mathrm{d}\tilde{p}}{\mathrm{d}r}, \qquad \mathrm{i}R(\omega - Uk)\tilde{w} = \frac{\mathrm{i}m}{r}\tilde{p}, \qquad \mathrm{i}R(\omega - Uk)\tilde{u} = \mathrm{i}k\tilde{p} - R\frac{\mathrm{d}U}{\mathrm{d}r}\tilde{v},$$

$$\mathrm{i}(\omega - Uk)\tilde{p} = \frac{1}{R}\left(\mathrm{i}(\omega - Uk)\tilde{\rho} + \frac{\mathrm{d}R}{\mathrm{d}r}\tilde{v}\right),$$

which, after eliminating every acoustic variable but $\tilde{p}$, gives the Pridmore-Brown [16] equation in cylindrical form

$$\tilde{p}'' + \left(\frac{1}{r} + \frac{2kU'}{\omega - Uk} - \frac{R'}{R}\right)\tilde{p}' + \left(R(\omega - Uk)^2 - k^2 - \frac{m^2}{r^2}\right)\tilde{p} = 0, \tag{4}$$

where a prime denotes $\mathrm{d}/\mathrm{d}r$. The radial velocity $\tilde{v}$, needed for the boundary condition, is given by $\tilde{v} = \mathrm{i}\tilde{p}'/(R(\omega - Uk))$.

The boundary condition to be applied at $r = 0$ is to require $\tilde{p}$ to be regular. The boundary condition to be applied at $r = 1$ is $\tilde{p}/\tilde{v} = Z$, giving

$$\mathrm{i}\omega Z\tilde{p}' - \omega^2 R(1)\tilde{p} = 0.$$

The Myers boundary condition (eq. 1), under the assumption that the mean-flow is constant apart from an infinitely-thin boundary layer at $r = 1$, predicts that

$$\mathrm{i}\omega Z\tilde{p}' - \big(\omega - Mk\big)^2\tilde{p} = 0. \tag{5}$$

It is this boundary condition that we propose to modify here.

III. Derivation of the modified boundary condition

In this section, we consider a thin boundary layer about $r = 1$ of typical width δ, outside which the mean flow is uniform, so that $U(r) = M$ and $R(r) = 1$ for the majority of the flow. We will consider the limit of small δ with all other parameters remaining $O(1)$ independent of δ. The asymptotic procedure used here is similar to that of [24], except that [24] worked with an expansion of $\log(\tilde{p})$, while here we will work directly with the physical variables $\tilde{p}$ and $\tilde{v}$, and that here we allow for nonconstant density $R(r)$ (and hence nonconstant temperature) within the boundary layer.

If the boundary layer did not exist, the solution to eq. (4) regular at $r = 0$ would be

$$\tilde{p}_o(r) = E\mathrm{J}_m(\alpha r) \qquad \text{where} \qquad \alpha^2 = (\omega - Mk)^2 - k^2. \tag{6}$$

Accounting for the boundary layer to first order in δ, the asymptotics (the details of which are given in appendix A) give a composite asymptotic solution of

$$\tilde{p}(r) = E\mathrm{J}_m(\alpha r) - \alpha E\mathrm{J}_m'(\alpha)\int_0^r 1 - \frac{\big(\omega - U(r)k\big)^2 R(r)}{(\omega - Mk)^2}\,\mathrm{d}r + O(\delta^2). \tag{7}$$

This shows very good agreement with the numerics, as is shown in section §IV. However, in order to derive the correction to the Myers boundary condition to $O(\delta)$, it turns out to be necessary to derive the pressure within the boundary layer to $O(\delta^2)$. This is again done in appendix A, with the result that, evaluated at the wall lining $r = 1$,

$$\begin{aligned}
\tilde{p}(1) &= E\mathrm{J}_m(\alpha) - \alpha E\mathrm{J}_m'(\alpha)\delta I_0 + O(\delta^2),\\
\tilde{v}(1) &= \frac{\mathrm{i}\big(\omega - U(1)k\big)}{(\omega - Mk)^2}\left[\alpha E\mathrm{J}_m'(\alpha) - \big(k^2 + m^2\big)\delta I_1 E\mathrm{J}_m(\alpha) + O(\delta^2)\right],
\end{aligned}$$

with

$$\delta I_0 = \int_0^1 1 - \frac{\big(\omega - U(r)k\big)^2 R(r)}{(\omega - Mk)^2}\,\mathrm{d}r, \qquad \delta I_1 = \int_0^1 1 - \frac{(\omega - Mk)^2}{\big(\omega - U(r)k\big)^2 R(r)}\,\mathrm{d}r. \tag{8}$$

Note that, since the integrands of δI_0 and δI_1 are identically zero outside the boundary layer, δI_0 and δI_1 are indeed both of order δ.

By stipulating that $Z = \tilde{p}(1)/\tilde{v}(1)$, we arrive at the dispersion relation

$$\mathrm{i}\big(\omega - U(1)k\big)Z\big[\alpha\mathrm{J}_m'(\alpha) - \big(k^2 + m^2\big)\delta I_1\mathrm{J}_m(\alpha)\big] - (\omega - Mk)^2\big[\mathrm{J}_m(\alpha) - \alpha\mathrm{J}_m'(\alpha)\delta I_0\big] = 0.$$

Our generalization of the Myers boundary condition is finally arrived at by substituting $\tilde{p}_o = E\mathrm{J}_m(\alpha)$ and $\tilde{v}_o = \mathrm{i}\alpha E\mathrm{J}_m'(\alpha)/(\omega - Mk)$, meaning that $\tilde{p}_o$ and $\tilde{v}_o$ are the pressure and normal velocity at the wall obtained by ignoring the boundary layer. This gives the boundary condition

$$Z\left[\tilde{v}_o + \frac{k^2 + m^2}{\mathrm{i}(\omega - Mk)}\delta I_1\tilde{p}_o\right] = \frac{\omega - Mk}{\omega - U(1)k}\left[\tilde{p}_o + \mathrm{i}(\omega - Mk)\delta I_0\tilde{v}_o\right]. \tag{9}$$

This is the main result of the paper. For most purposes, no slip implies that $U(1) = 0$, although setting $U(1) \neq 0$ in order to model surface roughness [as in 26] is also possible.

A. Interpretation of the δI_0 term

At least part of the modified boundary condition (eq. 9) may be interpreted physically. Let us first define the boundary layer mass, momentum, and kinetic energy thicknesses of the boundary layer to be

$$\delta_{\text{mass}} = \int_0^1 1 - R(r)\,\mathrm{d}r, \qquad \delta_{\text{mom}} = \int_0^1 1 - \frac{R(r)U(r)}{M}\,\mathrm{d}r, \qquad \delta_{\text{ke}} = \int_0^1 1 - \frac{R(r)U(r)^2}{M^2}\,\mathrm{d}r.$$

Multiplying out δI_0 gives

$$\delta I_0 = \frac{1}{(\omega - Mk)^2}\left(\omega^2\delta_{\text{mass}} - 2\omega k M\delta_{\text{mom}} + k^2M^2\delta_{\text{ke}}\right),$$

which, when substituted into eq. (9) gives the boundary condition

$$\tilde{v}_o\Big[\mathrm{i}\big(\omega - U(1)k\big)Z + \omega^2\delta_{\text{mass}} - 2\omega k M\delta_{\text{mom}} + k^2M^2\delta_{\text{ke}}\Big] = \tilde{p}_o\Big[\mathrm{i}(\omega - Mk) - \frac{\omega - U(1)k}{\omega - Mk}Z\big(k^2 + m^2\big)\delta I_1\Big].$$

The δI_0 term may therefore be thought of as modifying the impedance seen by the acoustics from the actual impedance of the boundary, Z, to an effective impedance Z_{mod} given by

$$\mathrm{i}\big(\omega - U(1)k\big)Z_{\text{mod}} = \mathrm{i}\big(\omega - U(1)k\big)Z + \omega^2\delta_{\text{mass}} - 2\omega k M\delta_{\text{mom}} + k^2M^2\delta_{\text{ke}}.$$

These terms have a physical interpretation. To see this, first consider a theoretical acoustic lining modelled as a flexible impermeable sheet, whose displacement η in the normal direction pointing out of the fluid is governed by

$$d\frac{\partial^2\eta}{\partial t^2} = \tilde{p} - K\eta - D\frac{\partial\eta}{\partial t} + T\frac{\partial^2\eta}{\partial x^2}. \tag{10}$$

Here, $\tilde{p}$ is a forcing term due to the acoustic pressure, d is a mass density, K is a spring constant, D is a damping constant, and T is an elastic tension in the sheet. Assuming η and $\tilde{p}$ to have $\exp\{\mathrm{i}\omega t - \mathrm{i}kx\}$ dependence gives

$$\mathrm{i}\big(\omega - U(1)k\big)Z = \frac{\tilde{p}}{\eta} = -\omega^2 d + \mathrm{i}\omega D + K + k^2T.$$

The modified impedance Z_{mod} based on this theoretical boundary model is therefore

$$\mathrm{i}\big(\omega - U(1)k\big)Z_{\text{mod}} = \frac{\tilde{p}}{\eta_{\text{mod}}} = -\omega^2(d - \delta_{\text{mass}}) + \mathrm{i}\omega D + K + k^2(T + M^2\delta_{\text{ke}}) - 2\omega k M\delta_{\text{mom}}, \tag{11}$$

which could be interpreted as a theoretical boundary with displacement η_{mod} and governing equation

$$(d - \delta_{\text{mass}})\left(\frac{\partial}{\partial t} - \frac{M\delta_{\text{mom}}}{d - \delta_{\text{mass}}}\frac{\partial}{\partial x}\right)^2\eta_{\text{mod}} = \tilde{p} - K\eta_{\text{mod}} - D\frac{\partial\eta_{\text{mod}}}{\partial t} + \left(T + M^2\left(\delta_{\text{ke}} + \frac{\delta_{\text{mom}}^2}{d - \delta_{\text{mass}}}\right)\right)\frac{\partial^2\eta_{\text{mod}}}{\partial x^2}. \tag{12}$$

The modified boundary layer term δI_0 therefore represents three physical effects:

1. The mass deficit δ_{mass} in the boundary layer causes the effective boundary seen by the acoustics to be lighter.
2. The momentum deficit $M\delta_{\text{mom}}$ in the boundary layer causes the advection (accounted for in the Myers boundary condition at a velocity M) to be corrected by the effective velocity deficit of the boundary layer, $M\delta_{\text{mom}}/(d - \delta_{\text{mass}})$.
3. The kinetic energy deficit $\frac{1}{2}M^2\delta_{\text{ke}}$ in the boundary layer, together with the momentum deficit, instantiate themselves as a tension along the boundary.

Note from eq. (11) that these effects become significant when k and ω are $O(\delta^{-1/2})$, and that in particular the effective tension becomes significant irrespective of ω when $k = O(\delta^{-1/2})$; these values of k and ω remain within the range of validity of the asymptotic derivation given in appendix A. The effect of tension along the boundary is particularly interesting, since it suggests there may be unforced travelling waves permitted along the boundary (which would be damped if $D \neq 0$). Note that if $T = 0$ the uncorrected boundary displacement (eq. 10) is local and does not support or prevent travelling waves, whereas the modified boundary (eq. 12) is always nonlocal and does support travelling waves (this effect was previously noted by Eversman [25]). In the limit $\delta \to 0$, this added tension disappears and, at fixed frequency, the wavenumbers for these travelling waves tend to infinity as $\delta \to 0$. This may well correspond to the illposedness and instability of the Myers boundary condition at arbitrarily short wavelengths [13], although this will not be pursued further here.

B. Interpretation of the δI_1 term

The δI_1 term is important, as it appears to be this term that is responsible for the wellposedness of the modified boundary condition. However, the interpretation of the δI_1 term is tricky, and no physical explanation of it is given here. It is worth noting that the presence a critical layer at $r = r^*$ (where r^* is possibly complex), for which $\omega - U(r^*)k = 0$, may have a significant affect on δI_1, especially if $|(1 - r^*)/\delta| \ll 1$. The case $r^* \in [0, 1]$ is explicitly excluded from our analysis, as in this case δI_1 may not be well defined; this is in line with previous analyses [e.g. 24]. For fixed ω, this exclusion causes a branch cut in the k-plane. This branch cut may be removable, as in the case of a constant-then-linear velocity profile given below, in which case δI_1 may be defined for $r^* \in [0, 1]$ by continuity.

Asymptotic values of δI_1 may be calculated, and it can be shown that, for $U(1) = 0$,

$$\begin{aligned}
\delta I_1 &\sim \int_0^1 1 - 1/R(r)\,\mathrm{d}r + \frac{2kM}{\omega}\int_0^1 1 - \frac{U(r)}{MR(r)}\,\mathrm{d}r && \text{for} \quad \omega/k \gg 1 \\
\delta I_1 &\sim \frac{-M^2 k}{\omega R(1) U'(1)} && \text{for} \quad \omega/k \ll 1
\end{aligned}$$

Moreover, for a constant density $R(r) \equiv 1$ and a constant-then-linear boundary layer profile,

$$U(r) = \begin{cases} M(1 - r)/\delta & (1 - r) < \delta \\ M & (1 - r) > \delta \end{cases}, \tag{13}$$

δI_1 may be directly calculated to give $\delta I_1 = \delta M k/\omega$.

C. The modified boundary condition in the time domain

In order to apply the new modified boundary condition in the time domain, eq. (9) needs some coercing into a suitable form. One possible way to do this would be to assume that the density is uniform and the boundary layer profile is constant-then-linear (eq. 13) with a given momentum thickness $\delta_{\mathrm{mom}} = \delta/2$. Then eq. (9) may be rearranged to give

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + M\frac{\partial}{\partial x}\right)\tilde{v} + 2M\delta_{\mathrm{mom}}\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2}\right)\tilde{p} = \left(\frac{\partial}{\partial t} + M\frac{\partial}{\partial x}\right)^2 \tilde{v}_b,$$

where $\tilde{v}_b = \tilde{p}/Z_{\mathrm{mod}}$ is the velocity of the boundary using a modified boundary model incorporating the δI_0 terms from §III.A above. For example, if the desired boundary model in the time domain were as given in eq. (10), then $\tilde{v}_b$ would be given by $\tilde{v}_b = \partial\eta_{\mathrm{mod}}/\partial t$ with η_{mod} satisfying eq. (12).

IV. Comparison with numerics

In order to validate the above asymptotics, the asymptotic solution will be compared with numerical solutions to the full Pridmore-Brown equation (4). The numerical solutions were generated using an implicit 12th order symmetric finite-difference discretization with unevenly spaced collocation points, so that the collocation points could be clustered within the boundary layer. Typically 8000 points were used, with at least 400 points within the boundary layer irrespective of the width of the boundary layer. The $r = 1$

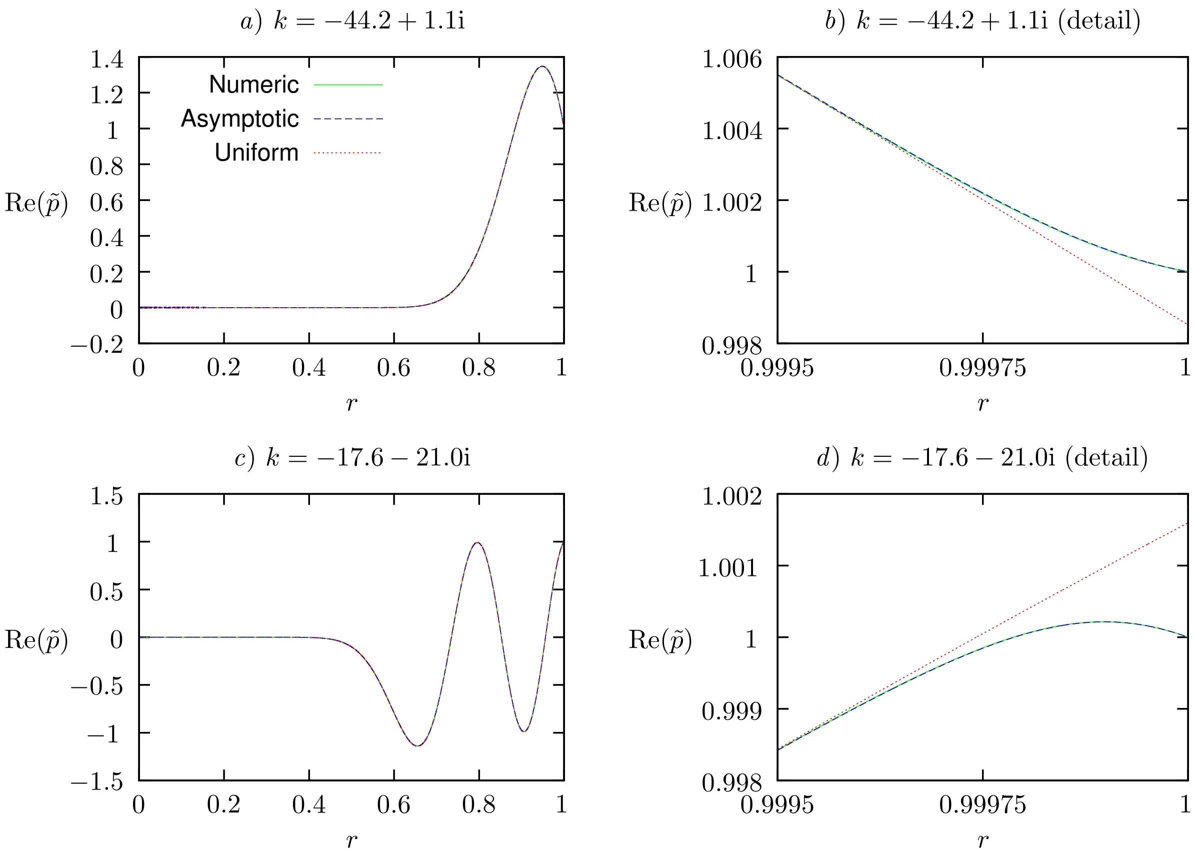

Figure 1. Plots of $\mathrm{Re}(\tilde{p}(r))$ against r, comparing the numerical solution to the Pridmore-Brown equation (4), the first-order asymptotic solution (eq. 7), and the uniform-flow solution (eq. 6). $\omega = 31$, $m = 24$, and $U(r)$ is given by eq. (14) with $M = 0.5$ and $\delta = 2 \times 10^{-4}$. The values of k used correspond to a cuton upstream-propagating mode (*a* and *b*) and a partially cutoff downstream-propagating mode (*c* and *d*) if the lining has impedance $Z = 2 + 0.6\mathrm{i}$.

boundary condition used was $\tilde{p}(1) = 1$; a Newton–Raphson iteration was then used to find roots of the dispersion relation $\tilde{p}/\tilde{v} = Z$.

The boundary layer profile used for most examples given here was chosen to be the tanh profile, as used by Rienstra & Vilenski [27],

$$U(r)/M = \tanh\left(\frac{1-r}{\delta}\right) + \big(1 - \tanh(1/\delta)\big)\left(\frac{1+\tanh(1/\delta)}{\delta} r + (1+r)\right)(1-r), \tag{14}$$

with constant density $R(r) \equiv 1$.

For this profile with $\delta = 2 \times 10^{-4}$, two comparisons between the numerical solution to eq. (4), the first-order-accurate composite solution (eq. 7) and the uniform solution are given in figure 1. Figure 2 gives a comparison between duct modes satisfying the impedance boundary condition using the numerics, the Myers boundary condition, and the modified boundary condition (eq. 9). Despite this being for such a thin boundary ($\delta = 2 \times 10^{-4}$), the highly cutoff modes and the surface mode in the upper-right quadrant are noticeably affected by the finite thickness of the boundary layer. The general trend that surface modes, more highly-attenuated modes and upstream-propagating modes are more affected by the boundary layer could be due to the nature of these modes, which oscillate significantly at the boundary, while modes which oscillate less on the boundary (such as the downstream-propagating cuton modes) see less of an effect.

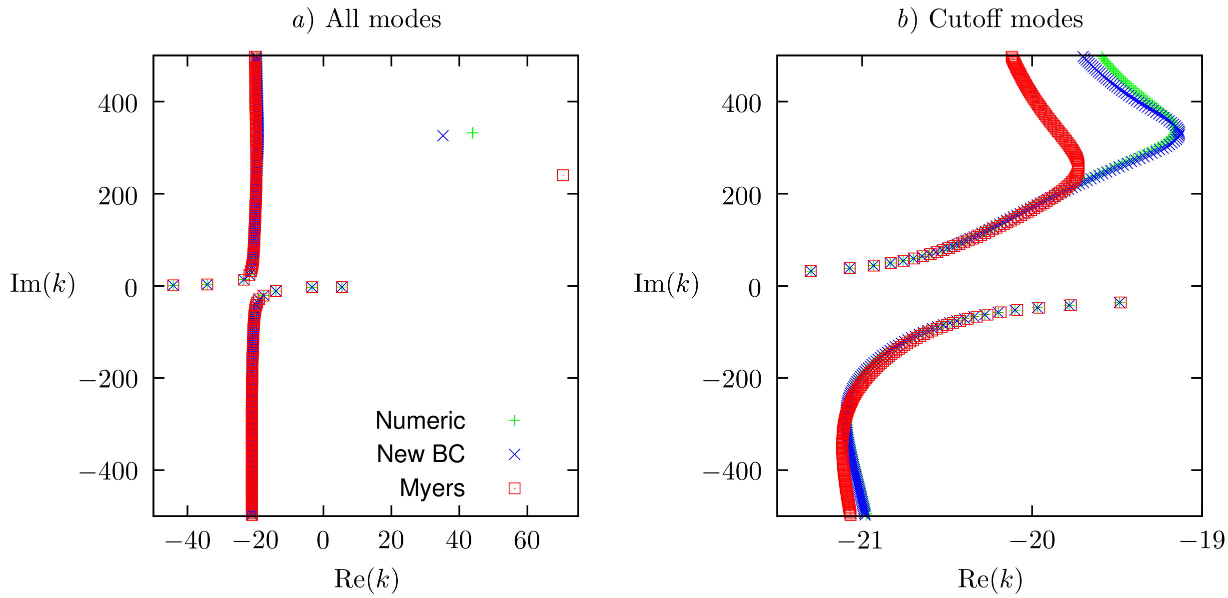

Figure 2. Comparison of axial wavenumbers k, plotted in the k-plane, for fully numerical solutions and solutions of the modified (eq. 9) and original (eq. 5) Myers boundary conditions. $\omega = 31$, $m = 24$, $Z = 2 + 0.6\mathrm{i}$, and $U(r)$ is given by eq. (14) with $M = 0.5$ and $\delta = 2 \times 10^{-4}$.

V. Stability and well-posedness

One major need for a generalization of the Myers boundary condition is that the Myers boundary condition is illposed [13], manifesting itself as numerical instability [9, 12], which we hope to alleviate using this modified boundary condition (eq. 9). Whatever the boundary condition at $r = 1$, it specifies either allowable values of ω if k is given, or allowable values of k if ω is given; these are here referred to as modes. For the problem to be wellposed, there should be a lower bound to $\mathrm{Im}\big(\omega(k)\big)$ for real k. For further explanation, see [13].

Figure 3 shows $\omega(k)$ as k is varied with $\mathrm{Im}(k) = 0$. The parameters used are the taken from Rienstra & Vilenski [27]. Note that for both the numerical solution and the modified boundary condition, $\mathrm{Im}\big(\omega(k)\big)$ is bounded below (say by -5), while the Myers solution is unbounded. Hence, the numerical and modified boundary condition solutions are wellposed. Since they are wellposed, we may apply the Briggs–Bers stability analysis [31, 32] to these cases. As there are real values of k with $\mathrm{Im}\big(\omega(k)\big) < 0$, there is a convective instability present. However, we must also look for absolute instabilities, where the system chooses its own preferred frequency and disturbances at that frequency grow exponentially in time. Absolute instability occurs for values of ω with $\mathrm{Im}(\omega) < 0$ for which two modes collide in the k-plane giving a double root. Only if this collision is between a mode originating for large $\mathrm{Im}(\omega)$ from the lower-half k-plane and one originating from the upper-half k-plane does this then signify an absolute instability [see 13, for details]. For this example, such a pinch does occur (as shown in figure 3), but for $\mathrm{Im}(\omega) > 0$, so that in this case no absolute instabilities are present. However, as the boundary layer thickness δ is reduced, this double root moves into the lower-half ω-plane and produces an absolute instability. The critical value of δ for which $\mathrm{Im}(\omega) = 0$ for this double root is found numerically to be 8.6×10^{-4} for the numerics, occurring at $\omega = 4.3$, $k = 19.8 + 95.3\mathrm{i}$, and $\delta = 9.7 \times 10^{-4}$ using the new modified boundary condition, occurring at $\omega = 3.7$ and $k = 22.8 + 82.1\mathrm{i}$.

For values of δ sufficiently large that there is no absolute instability, the system here is convectively unstable, with a downstream-propagating mode that has $\mathrm{Im}(k) > 0$. This can be seen by applying the Briggs–Bers [31, 32] criterion, which says that no convective instabilities are present provided $\mathrm{Im}(\omega)$ is sufficiently negative; i.e. below all $\omega(k)$ for real k. We may therefore ascertain the stability of modes by tracking them in the k-plane as ω is varied from sufficiently imaginary to real, with those that originate in the low-half k-plane propagating in the positive x-direction and those that originate in the upper-half k-plane propagating in the negative x-direction. This is done for $\mathrm{Re}(\omega) = 5$ and $\mathrm{Im}(\omega) \in [-5, 0]$ in figure 4.

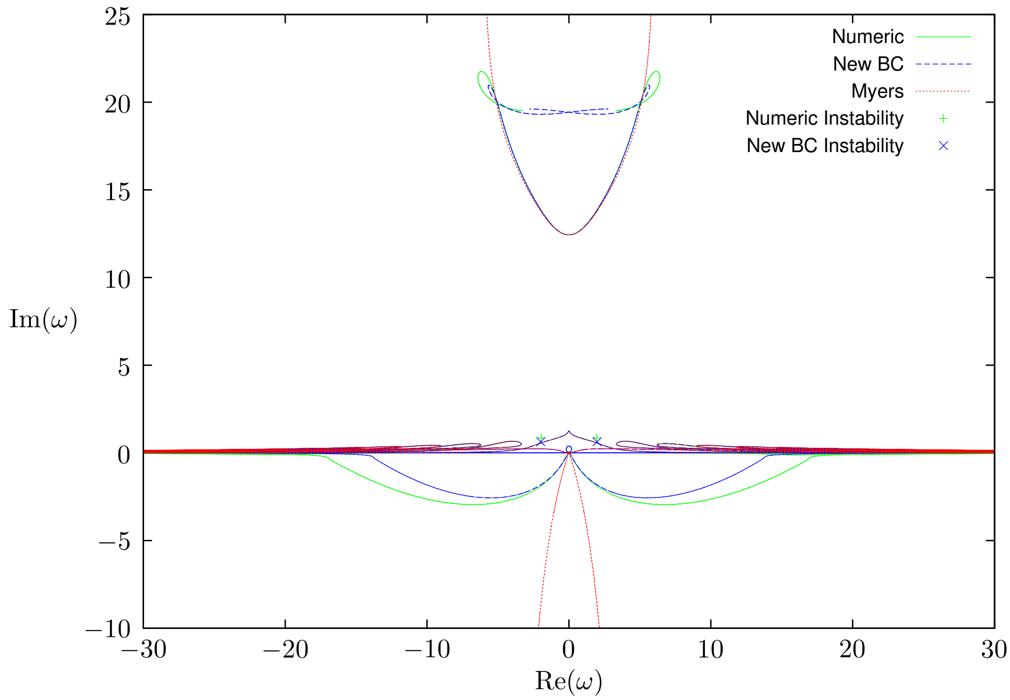

Figure 3. Plotted in the ω-plane are trajectories of $\omega(k)$ as k is varied with k real, for the numerically-calculated roots, solutions to the modified boundary condition (eq. 9), and the original Myers boundary condition (eq. 5). $\times$ and $+$ denote values of ω for which two k-roots coincide (a double root). $m = 0$, $Z = 3 + 0.15\mathrm{i}\omega - 1.15\mathrm{i}/\omega$, with $U(r)$ given by eq. (14) with $M = 0.5$ and $\delta = 2 \times 10^{-3}$.

All modes but one are seen to be stable. The unstable mode on the far right of figure 4 is the mode previously suggested by Rienstra [33] as a possible hydrodynamic instability mode. Note that the original Myers boundary condition fails to correctly predict the behaviour of this mode, while the modified boundary condition gives an accurate prediction.

There are several different models for the impedance Z. For single-frequency simulations a common assumption is that Z is constant. However, even in these situations any dependence of Z on ω would be important for stability. In order to investigate solutions for fixed Z, the same situation as figure 3 but with Z fixed at a constant value of $3 + 1.39\mathrm{i}$ (an unfeasible impedance in reality; see [34]) is shown in figure 5. This demonstrates that the new boundary condition also regularizes this problem even if the impedance is taken to be constant, although in this case an absolute instability is now present, as seen in the figure.

Recently, Rienstra & Darau [28] performed a similar analysis using a two-dimensional incompressible model, a constant-then-linear velocity profile (eq. 13) and a Helmholtz resonator model for the impedance, $Z = D + \mathrm{i}d\omega - \mathrm{i}\cot(\omega L)$. Nondimensionalizing their parameters under the assumption of a 0.2 meter diameter cylindrical duct (chosen to exaggerate any effect of the cylindrical geometry), the corresponding stability graph of $\omega(k)$ for real k is given in figure 6. In this case, Rienstra & Darau predict absolute instability for $\delta < 3.6 \times 10^{-4}$. Here, the modified boundary condition and the numerics both predict a critical value for δ of 3.7×10^{-4}, which is in good agreement with Rienstra & Darau considering their model was of a flat surface and incompressible flow, and here the duct is cylindrical and the flow compressible. Further comparisons with the model of Rienstra & Darau are given in §VI.

So far, all of these results have assumed an axisymmetric mode with azimuthal order $m = 0$. However, the new boundary condition multiplies the δI_1 term by $k^2 + m^2$, and so large values of m may lead to different behaviour. However, similar simulations to those shown here demonstrate that even in this case the problem is well posed.

The critical values for δ found to date are given in table 1. They show a good agreement between the modified boundary condition and the numerics, with very good agreement for thin boundary layers (as is to be expected from a correction that is asymptotically valid in the small-δ limit). In all these cases, the system was still convectively unstable for a boundary layer of width $\delta = 0.1$, though as δ is made progressively larger

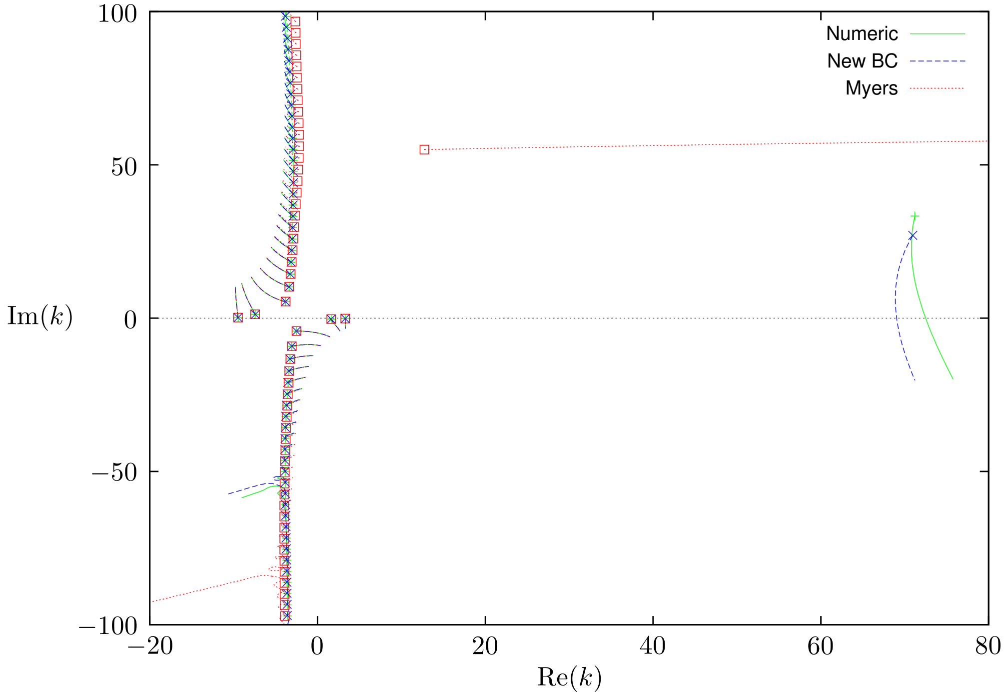

Figure 4. Trajectories of modes in the k-plane as ω is varied from 5 to $5 - 5\mathrm{i}$. The points are for $\omega = 5$. Since the modes on the far right cross the real k-axis as $\mathrm{Im}(\omega)$ is varied, they correspond to downstream-propagating instabilities. Parameters are (as in figure 3) $m = 0$, $Z = 3 + 0.15\mathrm{i}\omega - 1.15\mathrm{i}/\omega$, with $U(r)$ given by eq. (14) with $M = 0.5$ and $\delta = 2 \times 10^{-3}$.

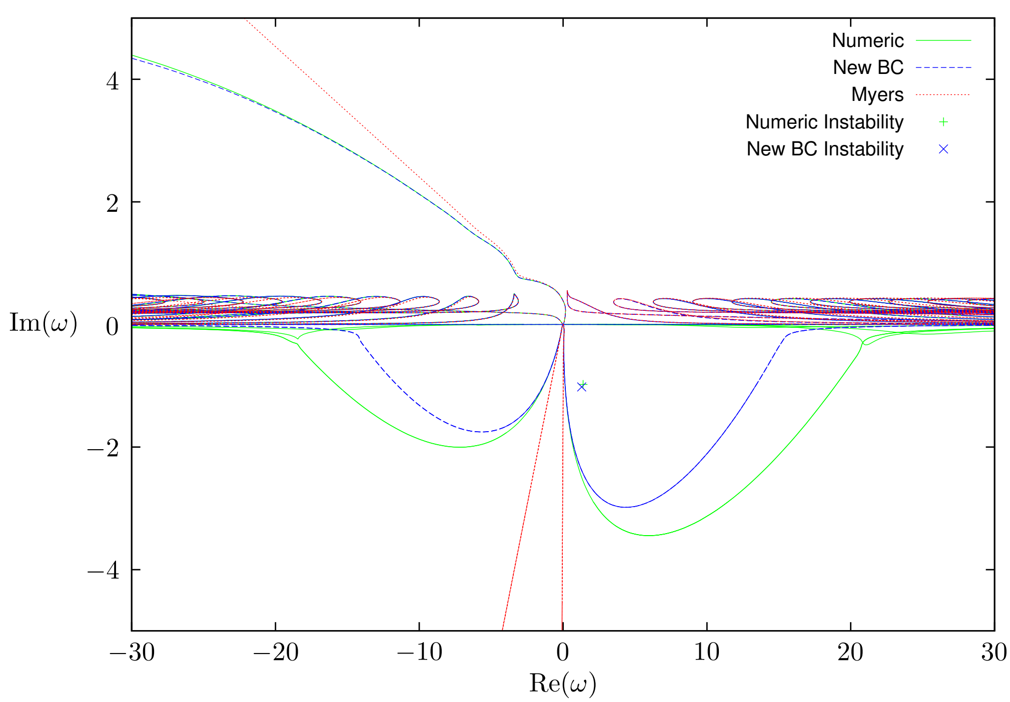

Figure 5. Trajectories of $\omega(k)$ as k is varied with k real, for the numerically-calculated roots, solutions to the modified boundary condition (eq. 9), and the original Myers boundary condition (eq. 5). $\times$ and $+$ denote values of ω for which two k-roots coincide (a double root). $m = 0$ and $Z = 3 + 1.39\mathrm{i}$, with $U(r)$ given by eq. (14) with $M = 0.5$ and $\delta = 2 \times 10^{-3}$.

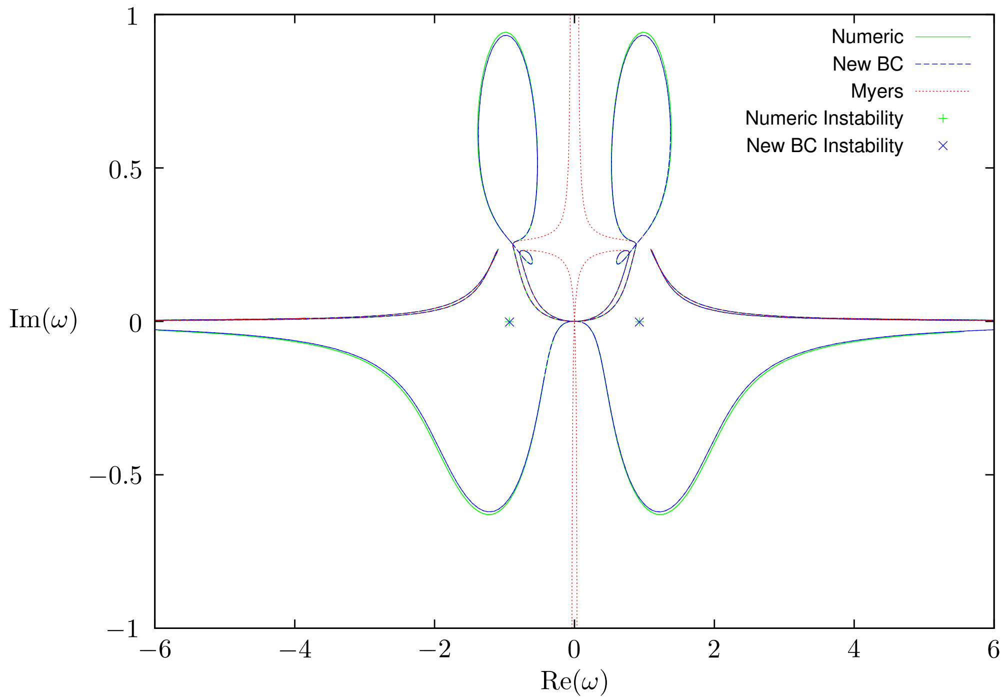

Figure 6. Trajectories of $\omega(k)$ as k is varied with k real for the numerically-calculated roots, solutions to the modified boundary condition (eq. 9), and the original Myers boundary condition (eq. 5). $\times$ and $+$ denote values of ω for which two k-roots coincide (a double root). $m = 0$ and $Z = 2 + 3.9\mathrm{i}\omega - \mathrm{i}\cot(0.3\omega)$, with $U(r)$ given by eq. (13) with $M = 0.176$ and $\delta = 3.6 \times 10^{-4}$.

$U(r)$ **profile**	M	m	Z	δ **numerical**	δ **asymptotic**
tanh	0.5	0	$3 + 0.15\mathrm{i}\omega - 1.15\mathrm{i}/\omega$	8.6×10^{-4}	9.7×10^{-4}
tanh	0.5	0	$3 + 1.39\mathrm{i}$	3.0×10^{-2}	8.2×10^{-2}
tanh	0.5	24	$2 + 0.8\mathrm{i}\omega - \mathrm{i}\cot(0.06\omega)$	1.7×10^{-3}	1.7×10^{-3}
linear	0.176	0	$2 + 3.9\mathrm{i}\omega - \mathrm{i}\cot(0.3\omega)$	3.7×10^{-4}	3.7×10^{-4}

Table 1. Table of critical boundary layer thicknesses (δ) for several different situations. For smaller values of δ the system is absolutely unstable. For larger values of δ the system is at most convectively unstable.

the frequency range for which convective instabilities can occur become smaller. For very small values of δ the solution approximates the Myers solution until k becomes sufficiently large that the $O(\delta)$ term in the modified boundary condition become important and the problem remains regularized. Setting $\delta = 0$ yields exactly the Myers boundary condition and the problem once again becomes illposed.

VI. Comparison with Rienstra & Darau's boundary condition

As mentioned previously, Rienstra & Darau [28] recently (and concurrently with the work presented here) analysed the stability of an incompressible two-dimensional shear layer over an acoustic lining. As part of this work, they proposed a modified Myers boundary condition [28, eq. 25] which, written in the notation used here, is

$$Z = \frac{\mathrm{i}(\omega - Mk)\tilde{p}_o - \big[2\delta_{\mathrm{mom}} Mk\omega\big(1 - \omega/(Mk)\big) - \delta_{\mathrm{ke}} M^2k^2/2\big]\tilde{v}_o}{\mathrm{i}\omega\tilde{v}_o - 2\delta_{\mathrm{mom}} k^2 \tilde{p}_o}. \tag{15}$$

Putting the modified Myers boundary condition given here (eq. 9) into a similar form for the same constant-then-linear boundary layer profile (eq. 13) with $R \equiv 1$ gives

$$Z = \frac{\mathrm{i}(\omega - Mk)\tilde{p}_o + \big[2\delta_{\mathrm{mom}} Mk\omega - \delta_{\mathrm{ke}} M^2k^2\big]\tilde{v}_o}{\mathrm{i}\omega\tilde{v}_o - 2\delta_{\mathrm{mom}}\big(k^2+m^2\big)\big(1-\omega/(Mk)\big)^{-1}\tilde{p}_o}. \tag{16}$$

Comparing eq. (15) and eq. (16) shows them to be very similar, although not identical, with both reducing to the unmodified Myers boundary condition in the limit $\delta \to 0$. More recently, Rienstra & Darau [29] pointed out that such boundary conditions are not unique, and that eq. (15) may be written for any $\tilde{\theta}$ as (again using the notation of this paper)

$$Z = \frac{\mathrm{i}(\omega - Mk)\tilde{p}_o + \big[2\delta_{\mathrm{mom}}(1-\tilde{\theta})\omega^2 - 2\delta_{\mathrm{mom}}(1-2\tilde{\theta})Mk\omega + \delta_{\mathrm{ke}}(1-3\tilde{\theta})M^2k^2/2\big]\tilde{v}_o}{\mathrm{i}\omega\tilde{v}_o - 2\delta_{\mathrm{mom}}k^2\tilde{p}_o - 2\mathrm{i}\omega\delta_{\mathrm{mom}}\tilde{\theta}\tilde{v}_o'}, \tag{17}$$

where $\tilde{v}_o'$ denotes the radial derivative of the outer uniform-flow solution $\tilde{v}_o(r)$ evaluated at $r = 1$. This equation is eq. (4.4) of [29]. In the 2D incompressible case considered by [28, 29] $\tilde{v}_o' = \mathrm{i}k^2\tilde{p}_o/(\omega - Mk)$, and therefore *in this case* eq. (17) and the boundary condition derived here (eq. 16) are identical for $\tilde{\theta} = 1$; hence, the boundary conditions of [28, 29] are asymptotically equivalent to eq. (16) for 2D incompressible perturbations. Rienstra & Darau [29] went on to propose $\tilde{\theta} = 1/3$ as a giving particularly pleasing results. However, the identity $\tilde{v}_o' = \mathrm{i}k^2\tilde{p}_o/(\omega - Mk)$ is only valid for 2D incompressible perturbations, with the equivalent here being $\tilde{v}_o' = -\tilde{v} - \mathrm{i}\big((\omega - Mk)^2 - k^2 - m^2\big)\tilde{p}_o/(\omega - Mk)$; therefore, the modified boundary conditions proposed in [28, eq. (25)] and [29, eq. (4.4)], corresponding to $\tilde{\theta} = 0$ and $\tilde{\theta} = 1/3$ in eq. (17) respectively, are not asymptotically equivalent to the modified boundary condition proposed here when applied to a compressible fluid.

It should be noted that eq. (16) is also not unique, as multiplying both numerator and denominator by $1 - 2\mathrm{i}\delta_{\mathrm{mom}}(1-\tilde{\theta})(\omega - Mk)\tilde{v}_o/\tilde{p}_o$ and using the identity $\tilde{v}_o/\tilde{p}_o = (\omega - Mk)/(\omega Z) + O(\delta)$ gives the asymptotically-equivalent expression for arbitrary $\tilde{\theta}$ (where $\tilde{\theta}$ is $O(1)$),

$$Z = \frac{\mathrm{i}(\omega - Mk)\tilde{p}_o + \big[2\delta_{\mathrm{mom}}(1-\tilde{\theta})\omega^2 - 2\delta_{\mathrm{mom}}(1-2\tilde{\theta})Mk\omega + \delta_{\mathrm{ke}}(1-3\tilde{\theta})M^2k^2/2\big]\tilde{v}_o}{\mathrm{i}\omega\tilde{v}_o - 2\delta_{\mathrm{mom}}\big(k^2+m^2\big)\tilde{p}_o + \dfrac{2\delta_{\mathrm{mom}}}{\omega(\omega - Mk)}\Big[\omega^2\big(k^2+m^2\big) + (\omega - Mk)^4(1-\tilde{\theta})/Z^2\Big]\tilde{p}_o}.$$

However, here we will continue with $\tilde{\theta} = 1$, for which this reduces to the originally-derived eq. (16), in order to avoid the term involving $(\omega - Mk)^4/Z^2$ in the denominator.

We now compare the boundary condition derived here (eq. 16) with the boundary condition of Rienstra & Darau (eq. 17) for $\tilde{\theta} = 0$ (as originally derived in [28]) and $\tilde{\theta} = 1/3$ (as proposed in [29]), as well as with the unmodified Myers boundary condition and the numerical solution, using a constant-then-linear boundary layer profile. Figure 7 gives such a comparison for the parameters used by Rienstra & Darau [28], for $\omega = 1$ and $m = 0$. The surface mode (shown in detail in figure 7*b*) is not correctly predicted by the unmodified Myers boundary condition, while all of the modified boundary conditions give reasonable estimates. The estimate of Rienstra & Darau for $\tilde{\theta} = 1/3$ is the most accurate in this case; this is expected, as the surface modes are inherently an incompressible phenomena [33], and therefore the flexibility of Rienstra & Darau's boundary condition for varying $\tilde{\theta}$ may be used without penalty. Note that their $\tilde{\theta} = 0$ estimate is comparably as accurate as the one for the boundary condition given here. For the downstream-propagating plane wave shown in figure 7*d*, however, the situation is reversed, with the boundary condition proposed here giving superb accuracy while both boundary conditions of Rienstra & Darau are less accurate than the unmodified Myers boundary condition. This too is expected, as it is for modes such as this that compressibility is important. Figure 7*c* indicates that the cut-off modes are almost equally well modelled by all of the modified boundary conditions (with, if anything, Rienstra & Darau's $\tilde{\theta} = 1/3$ being the most accurate in this case), with the unmodified Myers boundary condition being notably less accurate.

We now move to a more aeroacoustically relevant Helmholtz number of $\omega = 31$ and Mach number $M = 0.5$, with a thicker boundary layer of $\delta = 10^{-2}$ intended to be more indicative of a fully-developed turbulent boundary layer (and chosen so as to exaggerate any discrepancies between the exact and asymptotic predictions). Taking the lining to consist of Helmholtz resonators of depth $L = 35\,\mathrm{mm}$ with a facing sheet of mass reactance $d/R(1) = 20\,\mathrm{mm}$ (as used in [29]), and taking the duct to have radius $1.25\,\mathrm{m}$ gives a nondimensional impedance of $Z = 2 - 0.35\mathrm{i}$. We continue with $m = 0$ and the constant-then-linear

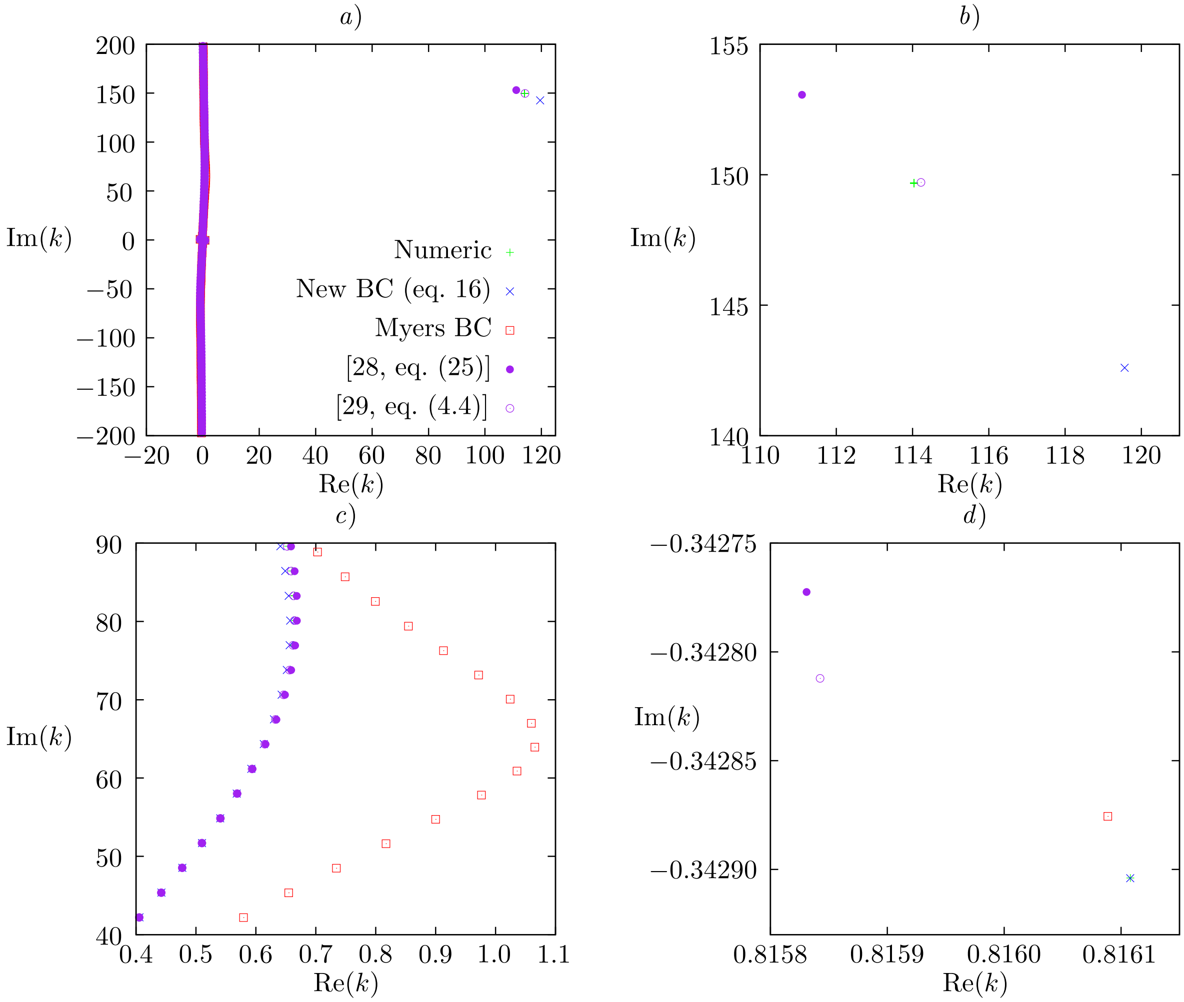

Figure 7. Comparison of the proposed boundary condition of Rienstra & Darau (eq. 17) with $\tilde{\theta} = 0$ [28, eq. (25)] or $\tilde{\theta} = 1/3$ [29, eq. (4.4)], the modified boundary condition proposed here (eq. 16), the unmodified Myers boundary condition (eq. 5), and the numerical solution, for a constant-then-linear velocity profile (eq. 13). Parameters are as for [28] with a 0.2 meter diameter cylindrical duct, giving $M = 0.176$, $\delta = 3.6 \times 10^{-4}$, and $Z = 2 + 3.9\mathrm{i}\omega - \mathrm{i}\cot(0.3\omega)$, with $\omega = 1$ and $m = 0$ (giving $Z = 2 + 0.667\mathrm{i}$).

boundary layer profile for a fair comparison with [28, 29]. For these parameters, a comparison of the various boundary conditions is shown in figure 8. Figure 8a shows the almost-propagating modes, and demonstrates that for upstream-propagating modes (shown in more detail in figure 8c) the modified boundary condition given here again correctly predicts the attenuation rate of these modes (given by $\mathrm{Im}(k)$), while either of the predictions of Rienstra & Darau are less accurate. Figure 8d shows a similar plot for the low-radial-order downstream-propagating modes, which in this case shows the modified boundary condition proposed here to give superb accuracy. The surface mode shown in the lower-right of figure 8d is only correctly predicted by the boundary condition proposed here, with neither of Rienstra & Darau's boundary conditions predicting its existence. Interestingly, for the upstream-propagating cutoff modes (shown in figure 8b) with approximately $50 \leq \mathrm{Im}(k) \leq 90$, none of the boundary conditions appears to accurately replicate the numerical solution. This situation looks to be similar to the situation where a surface mode is in close proximity to the cutoff acoustic modes, and indeed this may well be the case here. This is supported by the observation that this behaviour is highly dependent on the value of the impedance used.

The results given above are generally indicative of results obtained for a variety of impedances; for space reasons, not all results calculated are reproduced here. It is emphasized that the boundary condition proposed here (eq. 9) is capable of modelling three-dimensional perturbations with nonzero m, boundary layer profiles other than constant-then-linear, and nonuniform boundary layer densities. Since the boundary

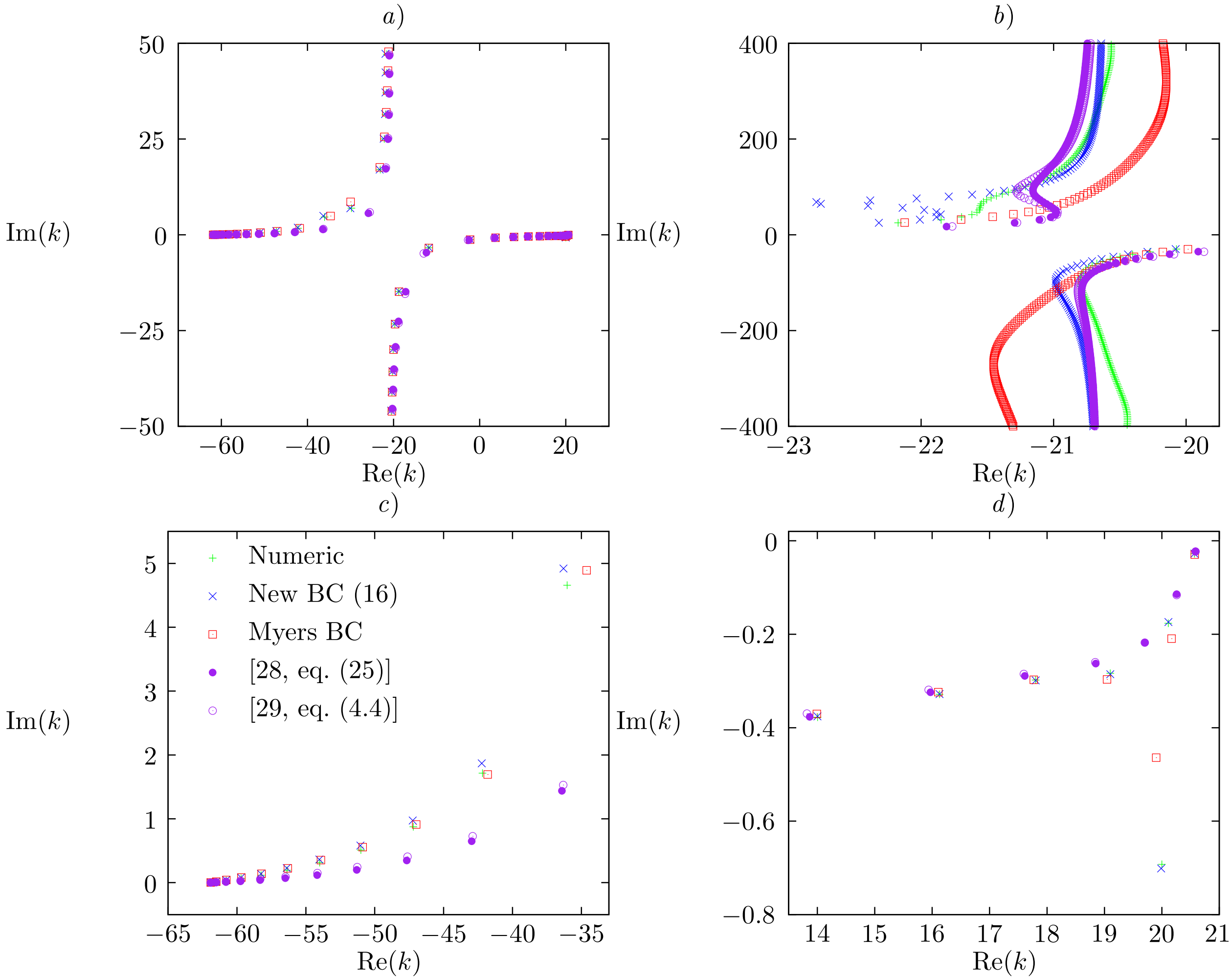

Figure 8. Comparison of the proposed boundary condition of Rienstra & Darau (eq. 17) with $\tilde{\theta} = 0$ [28, eq. (25)] or $\tilde{\theta} = 1/3$ [29, eq. (4.4)], the modified boundary condition proposed here (eq. 16), the unmodified Myers boundary condition (eq. 5), and the numerical solution, for a constant-then-linear velocity profile (eq. 13), for $M = 0.5$, $\delta = 10^{-2}$, $\omega = 31$, $Z = 2 + 0.016\mathrm{i}\omega - \mathrm{i}\cot(0.028\omega) = 2 - 0.35\mathrm{i}$ and $m = 0$.

conditions of Rienstra & Darau [28, 29] were derived assuming two-dimensional perturbations to a constant-then-linear uniform-density boundary layer, for nonzero m or for nonlinear boundary layer profiles the boundary condition proposed here is expected to give even better agreement with numerical solutions than the boundary conditions of [28, 29], although this is not investigated further here.

VII. Conclusion

The main conclusion of this paper is that the new modified Myers boundary condition, derived asymptotically for thin boundary layers and given in eq. (9) (with δI_0 and δI_1 given in eq. (8)), solves the illposedness problem associated with the standard Myers boundary condition. It can therefore be expected to alleviate the numerical instabilities associated with simulations using the Myers boundary condition in the time domain (applied, for example, as suggested in §III.C), as well as allowing a rigorous stability analysis (which is helpful for numerical simulations in the frequency domain).

The stability analyses conducted on the examples given here indicate that these examples are absolutely unstable for sufficiently thin boundary layers, and are convectively unstable otherwise. The only unstable mode found is the one predicted as being a hydrodynamic instability by Rienstra [33].

The new terms in the boundary layer are encapsulated within the δI_0 and δI_1 terms, which are integrals over the boundary layer and are valid for arbitrary boundary layer profiles. The δI_0 term may be calculated knowing only the mass, momentum, and kinetic energy thicknesses of the boundary layer. These three terms can be physically interpreted as a change in boundary mass, change in convected boundary speed, and a tension along the boundary. The δI_1 term is more difficult to classify, but expressions are given in §III.B in the high- and low-frequency limits, and an exact expression is given for δI_1 for a constant-then-linear boundary layer profile. It is the δI_1 term that is responsible for regularizing the Myers boundary condition, though it should be stressed that both terms are important in order that the new boundary condition be asymptotically accurate. The δI_1 term is also strongly linked to the presence of a critical layer at $r = r^*$, for which $\omega - U(r^*)k = 0$. The case $r^* \in [0, 1]$ is explicitly excluded from our analysis, leading to a branch cut in the k-plane (for ω considered fixed). For certain profiles, this branch cut may be removable, as is the case for a constant density and constant-then-linear boundary layer profile given in eq. (13), while for other profiles it may not be removable, possibly leading to algebraic growth [35]. The true nature and significance of the critical layer remains uncertain, and appears to depend strongly on the boundary layer profile [see, e.g. 36, 37].

The asymptotics have been derived under the assumption that $O(\delta^2)$ quantities may be neglected in the boundary condition; this is valid provided k, m and ω remain $O(1)$, or indeed provided they remain $o(1/\delta)$. For example, once k becomes sufficiently large that a wavelength becomes comparable to the boundary layer thickness, the asymptotics described in this paper can be expected to be inaccurate, although the modified boundary condition will remain well posed. In light of this, it is interesting to note that the proposed boundary condition appears to remain accurate for the majority of modes for the parameters $\omega = 31$ and $\delta = 0.01$, as used in figure 8, where it might have been argued a priori that $\omega = O(1/\delta)$. It should also be noted that the unmodified Myers boundary condition was derived under the assumption that $O(\delta)$ quantities may be neglected, and therefore the range of validity of the modified Myers boundary condition proposed here is greater than that of the unmodified Myers boundary condition.

While the derivation here has been for a straight cylindrical duct, exactly the same analysis is possible for a rectangular duct, leading to exactly the same modified boundary condition (a brief note on the mathematical derivation of this is given in appendix A.B). However, we have excluded curved ducts with non-parallel flows, for which the final term in the Myers boundary condition (eq. 1) is nonzero. Whether this term can be incorporated into a similar modified boundary condition, or indeed whether or not this term is correct for a permeable lining [11, see, e.g.], is not considered here.

As well as regularizing the Myers boundary condition, the new boundary condition may be used to gain insight into the effect of a boundary layer on the modal wavenumbers. The new boundary condition has been shown to be at least as accurate as the Myers boundary condition in all cases considered, and significantly more accurate in the majority of cases, as shown in figures 2, 4, 7 and 8. Moreover, when compared with the boundary conditions of Rienstra & Darau [28, 29], the cut-on acoustic modes are found more accurately with the boundary condition presented here, as seen in figures 7 and 8. The boundary condition of Rienstra & Darau [29] for $\tilde{\theta} = 1/3$ is shown to be more accurate for one surface mode shown in figure 7*b*; however, the $\tilde{\theta} = 1/3$ boundary condition fails to predict the existence of a different type of surface mode shown in figure 8*d*. The modified boundary condition presented here also has the advantage of being usable for three-dimensional perturbations for which $m \neq 0$, for nonlinear boundary layer velocity profiles, for nonconstant boundary layer temperature/density profiles, and for cylindrical as well as rectangular ducts.

For all examples considered here, $R(r) \equiv 1$, and so $\delta_{\mathrm{mass}} = 0$. However, cases when $\delta_{\mathrm{mass}} \neq 0$ are both realizable in practice and potentially interesting mathematically, especially for $\omega \gg 1$, since δI_0 includes an $\omega^2 \delta_{\mathrm{mass}}$ term. Such situations will be investigated further in future.

A. Derivation of the boundary layer asymptotics

In this appendix, we derive the acoustic field within a thin boundary layer of width δ correct to $O(\delta^2)$. For those not interested in the derivation, a summary of the major results derived here is given in the final paragraph of this appendix.

Our governing equation is the cylindrical form of the Pridmore-Brown equation (eq. 4) for the acoustic pressure in a sheared flow,

$$\tilde{p}'' + \left(\frac{1}{r} + \frac{2kU'}{\omega - Uk} - \frac{R'}{R}\right)\tilde{p}' + \left(R(\omega - Uk)^2 - k^2 - \frac{m^2}{r^2}\right)\tilde{p} = 0, \tag{4}$$

with the radial velocity given by $\tilde{v} = \mathrm{i}\tilde{p}'/(R(\omega - Uk))$. Here, $U(r)$ is the mean-flow velocity and $R(r)$ is the mean-flow density, nondimensionalized so that $U(0) = M$, the Mach number, and $R(0) = 1$. We consider a thin boundary layer about $r = 1$ of typical width δ, outside which the mean flow is uniform, so that $U(r) = M$ and $R(r) = 1$ for r outside the boundary layer.

If the boundary layer did not exist, the solution to eq. (4) would be $\tilde{p}(r) = \tilde{p}_o(r)$,

$$\tilde{p}_o(r) = E\mathrm{J}_m(\alpha r) \qquad\qquad \alpha^2 = (\omega - Mk)^2 - k^2.$$

Expanding this for $r = 1 - \delta y$, and utilizing Bessel's equation to write J_m'' in terms of J_m and J_m', gives

$$\tilde{p}_o(1 - \delta y) = E\mathrm{J}_m(\alpha) - \delta y \alpha E\mathrm{J}_m'(\alpha) - \tfrac{1}{2}\delta^2 y^2 E\Big[\alpha \mathrm{J}_m'(\alpha) + \big(\alpha^2 - m^2\big)\mathrm{J}_m(\alpha)\Big] + O(\delta^3). \tag{18}$$

This is our outer expansion, that the inner expansion within the boundary layer must match with.

To consider the boundary layer, it is helpful to first rearrange the governing equation to give

$$\left(\frac{r\tilde{p}'}{(\omega - Uk)^2 R}\right)' + \left(r - \frac{k^2 r + m^2/r}{(\omega - Uk)^2 R}\right)\tilde{p} = 0. \tag{19}$$

Substituting $r = 1 - \delta y$ into eq. (19) and using a subscript y to denote $\mathrm{d}/\mathrm{d}y$ gives

$$\left(\frac{\tilde{p}_y}{(\omega - Uk)^2 R}\right)_y = \delta\left(\frac{y\tilde{p}_y}{(\omega - Uk)^2 R}\right)_y - \delta^2\left(1 - \frac{k^2 + m^2}{(\omega - Uk)^2 R}\right)\tilde{p} + O(\delta^3).$$

In a slight but obvious abuse of notation, $U(y)$ is used to represent $U(r)$ with $r = 1 - \delta y$, and similarly for R. We pose the solution $\tilde{p} = \tilde{p}_0 + \delta\tilde{p}_1 + \delta^2\tilde{p}_2 + O(\delta^3)$. To leading order,

$$\tilde{p}_0 = A_0 + B_0 y - B_0 \int_0^y 1 - \frac{\big(\omega - U(y')k\big)^2 R(y')}{(\omega - Mk)^2}\,\mathrm{d}y'.$$

Matching with the outer solution (eq. 18) for large y at leading order gives $A_0 = E\mathrm{J}_m(\alpha)$ and $B_0 = 0$. Since $B_0 = 0$, at first order we similarly find

$$\tilde{p}_1 = A_1 + B_1 y - B_1 \int_0^y 1 - \frac{\big(\omega - U(y')k\big)^2 R(y')}{(\omega - Mk)^2}\,\mathrm{d}y',$$

while matching with the outer solution (eq. 18) for large y gives $B_1 = -\alpha E\mathrm{J}_m'(\alpha)$ and $A_1 = B_1 I_0$, with

$$I_0 = \int_0^\infty 1 - \frac{\big(\omega - U(y)k\big)^2 R(y)}{(\omega - Mk)^2}\,\mathrm{d}y.$$

At second order we find

$$\begin{aligned}\tilde{p}_2 = A_2 + B_2 \int_0^y \frac{\big(\omega - U(y')k\big)^2 R(y')}{(\omega - Mk)^2}\,\mathrm{d}y' + B_1 \int_0^y y' \frac{\big(\omega - U(y')k\big)^2 R(y')}{(\omega - Mk)^2}\,\mathrm{d}y' \\ - A_0 \int_0^y \big(\omega - U(y')k\big)^2 R(y') \int_0^{y'} 1 - \frac{k^2 + m^2}{\big(\omega - U(y'')k\big)^2 R(y'')}\,\mathrm{d}y''\,\mathrm{d}y'.\end{aligned}$$

Rewriting this in terms of bounded integrals to aid matching with the outer solution gives

$$\begin{aligned}\tilde{p}_2 = A_2 + B_2 y - B_2 \int_0^y 1 - \frac{\big(\omega - U(y')k\big)^2 R(y')}{(\omega - Mk)^2}\,\mathrm{d}y' + B_1 \int_0^y y' \left(\frac{\big(\omega - U(y')k\big)^2 R(y')}{(\omega - Mk)^2} - 1\right)\mathrm{d}y' + \tfrac{1}{2}B_1 y^2 \\ - A_0 \int_0^y \left(\frac{\big(\omega - U(y')k\big)^2 R(y')}{(\omega - Mk)^2} - 1\right)\int_0^{y'} (\omega - Mk)^2 - \big(k^2 + m^2\big)\frac{(\omega - Mk)^2}{\big(\omega - U(y'')k\big)^2 R(y'')}\,\mathrm{d}y''\,\mathrm{d}y' \\ - A_0\big(k^2 + m^2\big)\int_0^y \left(\int_0^{y'} 1 - \frac{(\omega - Mk)^2}{\big(\omega - U(y'')k\big)^2 R(y'')}\,\mathrm{d}y'' - I_1\right)\mathrm{d}y' \\ - A_0 \tfrac{1}{2}\Big((\omega - Mk)^2 - k^2 - m^2\Big)y^2 - A_0 I_1 (k^2 + m^2) y,\end{aligned}$$

where the integrals are bounded provided we define

$$I_1 = \int_0^\infty 1 - \frac{(\omega - Mk)^2}{\big(\omega - U(y)k\big)^2 R(y)}\,\mathrm{d}y$$

We now match this to the outer solution (eq. 18) for large y by equating powers of y. The y^2 terms match identically using our definitions of A_0 and B_1 above. Matching the y terms gives $B_2 = A_0 I_1(k^2+m^2)$, while matching the constant terms gives

$$\begin{aligned}A_2 = B_2 I_0 + B_1 \int_0^\infty y\left(1 - \frac{\big(\omega - U(y)k\big)^2 R(y)}{(\omega - Mk)^2}\right)\mathrm{d}y \\ + A_0 \int_0^\infty \left(\frac{\big(\omega - U(y)k\big)^2 R(y)}{(\omega - Mk)^2} - 1\right)\int_0^y (\omega - Mk)^2 - \big(k^2+m^2\big)\frac{(\omega - Mk)^2}{\big(\omega - U(y')^2\big)R(y')}\,\mathrm{d}y'\mathrm{d}y \\ + A_0(k^2+m^2)\int_0^\infty \left(\int_0^y 1 - \frac{(\omega - Mk)^2}{\big(\omega - U(y')^2\big)R(y')}\,\mathrm{d}y' - I_1\right)\mathrm{d}y.\end{aligned}$$

Fortunately, for the modified Myers boundary condition we do not need to evaluate A_2!

A. Summary

Now that we have the inner solution through the boundary layer, we may calculate $\tilde{p}$ and $\tilde{v}$ at the wall $r = 1$, $y = 0$. Using $\tilde{v} = \mathrm{i}\tilde{p}'/(R(\omega - Uk))$, we find that

$$\begin{aligned}\tilde{p}(1) &= E\mathrm{J}_m(\alpha) - \alpha E\mathrm{J}_m'(\alpha)\delta I_0 + O(\delta^2), \\ \tilde{v}(1) &= \frac{\mathrm{i}\big(\omega - U(1)k\big)}{(\omega - Mk)^2}\left[\alpha E\mathrm{J}_m'(\alpha) - \big(k^2+m^2\big)\delta I_1 E\mathrm{J}_m(\alpha) + O(\delta^2)\right],\end{aligned}$$

where we may rewrite δI_0 and δI_1 in terms of integrals over r as

$$\delta I_0 = \int_0^1 1 - \frac{\big(\omega - U(r)k\big)^2 R(r)}{(\omega - Mk)^2}\,\mathrm{d}r, \qquad \delta I_1 = \int_0^1 1 - \frac{(\omega - Mk)^2}{\big(\omega - U(r)k\big)^2 R(r)}\,\mathrm{d}r.$$

To first order, we may form the composite solution for the pressure,

$$\tilde{p}_c(r) = \mathrm{J}_m(\alpha r) - \alpha\mathrm{J}_m'(\alpha)\int_0^r 1 - \frac{\big(\omega - U(r)k\big)^2 R(r)}{(\omega - Mk)^2}\,\mathrm{d}r. + O(\delta^2).$$

B. Derivation for a rectangular duct

For a rectangular duct, we take a solutions of the form $\tilde{p} = \tilde{p}(r)\exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}mz\}$, where m is no longer restricted to taking integer values, z is the coordinate perpendicular to the flow in the plane of the liner, and "r" is the coordinate normal to the liner. The equivalent of the Pridmore-Brown eq. (4) in this case is

$$\tilde{p}'' + \left(\frac{2kU'}{\omega - Uk} - \frac{R'}{R}\right)\tilde{p}' + \big(R(\omega - Uk)^2 - k^2 - m^2\big)\tilde{p} = 0.$$

Using exactly the same procedure as above leads to exactly the same result as derived above, although the algebra is slightly different since the $1/r$ geometric term is not present in this case.

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